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ABSTRACT

LYONS, DAVID MICHAEL. A Moving Boundary Problem Modelling Diffusion with Nonlinear Absorption. (Under the direction of ROBERT H. MARTIN, JR).

A moving boundary problem involving a parabolic differential equation with a nonlinear absorption function f and with a constant coefficient of diffusion is analyzed. The function f is nonincreasing, $f(0) < 0$, and f is dependent only on $u = u(x,t;\ \Phi)$ where u is the function satisfying the differential equation, boundary conditions, and $u(x,0) = \Phi(x)$. Because of an application in a model of oxygen diffusion/absorption in living tissue, a nonnegative valued function u is sought as well as the unknown moving boundary $\gamma(t)$. Existence of a unique solution pair $\{u(x,t;\ \Phi),\gamma(t)\}$ is proved and, where u is positive, it is shown to be classical (u_{xx} and u_t exist and are Hölder continuous).

For α a fixed parameter appearing in the boundary condition at $x = 0$ and satisfying $0 \leq \alpha \leq \infty$, if $S^{\alpha}(t)\Phi \equiv u(\cdot,t;\ \Phi)$ for $t \geq 0$, then $S^{\alpha} = \{S^{\alpha}(t) : t \geq 0\}$ is determined to be a nonexpansive (C_o)-semigroup of nonlinear operators with a multivalued infinitesimal generator. Using the Hille product formula, the set of all nonincreasing (uniformly bounded) functions with compact support $[0,\rho]$ is shown to be invariant relative to S^{o} and S^{∞}. Additionally, the set of all (uniformly bounded) functions on $[0,\rho]$ which initially are nondecreasing and then are non-increasing is shown to be invariant relative to S^{α} for each positive finite value of α.

Whenever $\gamma(t)$ is decreasing, its continuity is established using backward uniqueness results. For the case when $\gamma(t)$ is increasing, the joint continuity of u (in x and t) is used to prove that $\gamma(t)$ is continuous. Further, it is possible for the moving boundary to be delayed,

i.e. a fixed boundary can occur at $x = \rho$ until some positive time at which the moving boundary begins. Finally, for all cases except $\alpha = \infty$, the trivial solution is the unique critical point and the moving boundary reaches zero in finite time. When $\alpha = \infty$, there exists a unique non-trivial critical point which can not be reached in finite time.

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DAVID MICHAEL LYONS
MAJOR, USAF

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A MOVING BOUNDARY PROBLEM MODELLING DIFFUSION

WITH NONLINEAR ABSORPTION

by

DAVID MICHAEL LYONS

A thesis submitted to the Graduate Faculty of
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in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

DEPARTMENT OF MATHEMATICS

RALEIGH

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APPROVED BY:

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BIOGRAPHY

David Michael Lyons was born in Greensburg, Pennsylvania on March 6, 1947. He was raised near Williamsport in Lycoming County and graduated from Williamsport Area High School in 1965. After attending Lycoming College for one year, he received an appointment to the United States Air Force Academy where he majored in mathematics. In 1970 he graduated from the Academy with a Bachelor of Science degree and received a commission in the Air Force.

Since 1970 the author has been working in a variety of Air Force duty assignments. Initially, he attended graduate school at North Carolina State University and earned the degree of Master of Applied Mathematics in 1971. In 1972 he completed a year of pilot training near Phoenix, Arizona and subsequently flew the OV-10 observation aircraft in Southeast Asia. In 1973, he returned to the United States with an assignment as an instructor pilot in the T-37 primary jet trainer and served a tour in Del Rio, Texas until 1977. At this time he was assigned to the faculty of the U.S. Air Force Academy in the Department of Mathematical Sciences. In 1979 he departed the Academy for an Air Force sponsored doctoral program at North Carolina State University. After an upcoming two-year flying tour at Patrick Air Force Base, Florida, he anticipates returning to a faculty position at the Air Force Academy in 1984.

The author is married to the former Peggy Murray of New Orleans, Louisiana. Mrs. Lyons graduated from St. Mary's Dominican College in 1970 with a Bachelor of Arts degree in elementary education. The Lyons' have a daughter, Jennifer, born in July, 1977, and a son, Timothy, born in December, 1979.

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During the last three years, there were several people who influenced or assisted me and who deserve recognition. For his encouragement, guidance, and patience during all phases of my studies, I express very special thanks to Professor Robert H. Martin, Jr., Chairman of my Advisory Committee, for whom I have the greatest admiration. To the other members of my Advisory Committee, Professors Carl D. Meyer, Jr., Lung O. Chung, and Gary D. Faulkner, I extend my gratitude for their assistance during the course of study. I am also particularly appreciative of the efforts of my typist, Sharon Jones, whose diligence and attention to detail were commendable. Last and most important, I am deeply grateful to my wife, Peggy, and to my children for their understanding and sacrifices during the entire program of study.

For his friendship and valuable counsel over many years, I extend a special thank you to Professor Howard M. Nahikian. Further, I wish to thank my parents whose guidance and support during my early years will never be forgotten.

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1. INTRODUCTION AND PRELIMINARIES

This thesis centers on the following nonlinear parabolic partial differential equation:

$$du_{xx} + f(u) = u_t.$$

The function f is negative valued when u is nonnegative and so f can be considered as an absorption term. Additionally, $f(0) < 0$ and f is assumed to be nonincreasing on $[0,\infty)$. Finally, f is continuously differentiable and its derivative is Hölder continuous on $[0,\infty)$. As we will show, given these characteristics of the nonlinearity f, a so-called moving (or free) boundary occurs and is unknown a priori.

The genesis for this work is a paper by Crank/Gupta [2] in 1972 and a subsequent paper by Rogers [10] in 1977. Crank/Gupta modelled the diffusion and absorption of oxygen in tissue with the equation

$$dC_{yy} - m = C_\tau$$

where $C(y,\tau)$ was the concentration of oxygen at depth y and time τ and where m was a constant rate of absorption (consumption) of oxygen per unit volume of the tissue. Initially at $y = 0$ a concentration $C = C_o$ was maintained until steady-state was achieved. Once a steady state occurred (where $C_\tau = 0$), there existed y_o such that $C(y_o,\cdot) = 0$. Also at $y = y_o$, no diffusion occurred (since $C(y_o,\cdot) = 0$) so that $C_y(y_o,\cdot) = 0$. It followed then that the steady state solution was

$$E(y) = (m/2d)(y_o - y)^2$$

where

$$y_o = (2dC_o/m)^{\frac{1}{2}}.$$

Next the surface of the tissue was sealed off (presumably perfectly) so that $C_y(0,\cdot) = 0$. At this point it was still assumed that diffusion with absorption occurred in the tissue but at the boundary $y = 0$ there was a Neumann condition. Crank/Gupta and Rogers worked primarily on the initial-boundary value problem which was a model for oxygen concentrations after the surface of the tissue was sealed (i.e. a Neumann boundary condition at $y = 0$). To put the problem in non-dimensional form, they let

$$x = y/y_o, \quad t = d\tau/y_o^2, \quad \text{and} \quad c(x,t) = C(y,\tau)/2C_o.$$

Then the problem was to find a solution pair $\{c(x,t),\gamma(t)\}$ for $0 < t < T$ such that the moving boundary $\gamma(t)$ satisfied

$$\gamma(0) = 1, \quad \gamma(t) > 0 \quad \text{for} \quad 0 < t < T$$

and $c(x,t)$ was defined on the domain

$$\Omega \equiv \{(x,t)|0 < x < \gamma(t), 0 < t < T\}$$

and satisfied

$$\left.\begin{array}{l} c_{xx} - 1 = c_t \quad \text{on} \quad \Omega \\ c_x(0,t) = 0, \quad c(\gamma(t),t) = c_x(\gamma(t),t) = 0 \quad \text{for} \quad 0 < t < T \\ \lim_{t\to 0^+} c(x,t) = (1 - x)^2/2 \quad \text{for} \quad 0 < x < 1 \end{array}\right\} \quad (1.1)$$

Note that $(1 - x)^2/2$ is the non-dimensional form of the steady state solution, $E(y) = (m/2d)(y_o - y)^2$.

Crank/Gupta used numerical techniques on the problem and their data indicated that $c(x,t)$ of the solution pair to (1.1) reached zero in

finite time (and accordingly the moving boundary reached zero in finite time). In addition, Crank/Gupta assumed for small enough values of time that the moving boundary was actually fixed and therefore that their analytical approximation to $c(x,t)$ was suitably accurate. We will show that for an initial value as given in (1.1) that $\gamma(t)$ does move immediately and, in fact, is strictly decreasing thereafter.

On the other hand Rogers studied existence and uniqueness of solutions to (1.1). Because he sought a nonnegative function for the concentration $c(x,t)$, Rogers approximated (1.1) with the following problem:

$$\left.\begin{aligned} &\bar{c}_{xx} - a_\varepsilon(\bar{c}) = \bar{c}_t \quad \text{for} \quad 0 < x < 1, \quad 0 < t < T \\ &\bar{c}_x(0,t) = 0, \quad \bar{c}(1,t) = 0, \quad 0 < t < T \\ &\lim_{t\to 0^+} \bar{c}(x,t) = f_o(x,\varepsilon), \quad 0 < x < 1, \text{ where} \\ &\lim_{\varepsilon\to 0^+} f_o(x,\varepsilon) = (1-x)^2/2. \end{aligned}\right\} \tag{1.2}$$

(1.2) is a family of problems which is parameter dependent on ε. For numerical considerations, Rogers approximated the initial condition in (1.1) with $f_o(x,\varepsilon)$ but this bears no significance in our study. What is important is the term $a_\varepsilon(\bar{c})$ in (2.1) because $a_\varepsilon(0) = 0$ and thereby Rogers was able to approximate with the nonnegative valued function $\bar{c}(x,t)$. Also, Rogers showed that

$$\lim_{\varepsilon\to 0^+} \bar{c}(x,t) = c(x,t),$$

that is, the solutions of (1.2) converged to the concentration function needed in (1.1). Finally, Rogers proved that $c(x,t)$ in (1.1) satisfied

$$c_x \leq 0 \quad \text{and} \quad c_t \leq 0 \quad \text{for} \quad (x,t) \in (0,1) \times (0,T).$$

In this thesis, we will clarify as well as correct deficiencies in Rogers proofmaking. Further, we will generalize (1.1) to include a variety of boundary conditions and initial values. Last but equally important in our work, we analyze the oxygen model while oxygen is supplied at the surface.

It is worthwhile to note that, in physical situations involving absorption (consumption), the absorption rate often increases according to increases in concentration. Rogers mentioned this phenomena for oxygen concentrations near zero and it is also possible that the absorption rate in many situations is nowhere constant. Therefore, as mentioned earlier, we will consider a parabolic partial differential equation with a nonlinear absorption term $f(u)$ which is nonincreasing. We state the complete initial-boundary value problem to be analyzed in this thesis as

$$\left.\begin{array}{l} du_{xx} + f(u) = u_t, \quad 0 < x < \gamma(t), \quad t > 0 \\ u_x(0,t) = \alpha u(0,t), \quad u(\gamma(t),t) = u_x(\gamma(t),t) = 0, \quad t > 0 \\ u(x,0) = \Phi(x), \quad 0 < x < \rho \end{array}\right\} \tag{1.3}$$

where α is a fixed real number satisfying $0 \leq \alpha \leq \infty$. (Notation in (1.3) is like that of Meyer in [7].) Also we use the convention that for $\alpha = \infty$ the boundary condition at $x = 0$ is

$$u(0,t) = c_0 > 0.$$

Thus when $\alpha = \infty$, the boundary condition at $x = 0$ is Dirichlet and non-homogeneous. Last, $d > 0$ is the constant coefficient of diffusion.

In the case $\alpha = \infty$, we observe that (1.3) includes that phase of the oxygen model which is prior to the achievement of steady-state and during which a fixed concentration (c_o) is supplied at the surface $x = 0$. In comparison, when $\alpha = 0$ in (1.3), we include that phase of the oxygen model which occurs after the surface of the tissue is sealed off. Finally, since there may not be a perfect seal at the surface, we include the possibility of "leakage" in the oxygen model by assigning α a positive finite value.

In chapter 2 we prove a fundamental existence and uniqueness theorem for (1.3) which involves classical smoothness properties of the function $u(x,t)$. Chapter 3 covers the application of the product formula for semigroups and set invariance. In chapter 4 we prove that the moving boundary $\gamma(t)$ is continuous and that, in certain cases, $\gamma(t)$ is either strictly decreasing or strictly increasing. Finally, in chapter 5 we show that $u(x,t)$ in (1.3) reaches the zero function in finite time if $\alpha < \infty$ and that a steady-state (equilibrium) solution cannot be reached in finite time if $\alpha = \infty$.

To complete this introductory chapter, we show that the function $u(x,t)$ of the solution $\{u(x,t),\gamma(t)\}$ to (1.3) is not just the positive part of a solution to the following less-constrained parabolic problem:

$$\left.\begin{array}{ll} dw_{xx} + f(w) = w_t, \quad 0 < x < \rho, \quad t > 0 \\ w_x(0,t) = \alpha w(0,t), \quad w(\rho,t) = 0, \quad t > 0 \\ w(x,0) = \Phi(x), \quad 0 < x < \rho. \end{array}\right\} \tag{1.4}$$

As stated earlier, it is assumed that f in (1.4) is continuously differentiable on $[0,\infty)$.

<u>Lemma 1.1</u>. Suppose that $w(x,t)$ is the solution to (1.4) and that w_x satisfies

$$d(w_x)_{xx} + f'(w)w_x = (w_x)_t, \quad 0 < x < \rho, \quad t > 0.$$

Then, if w_x is positive at some time t, a positive maximum occurs either at $x = 0$ or at $x = \rho$.

<u>Proof</u>. We seek a contradiction by assuming that

$$\sup\{w_x e^{-\delta t}\} = w_x(x_o,t_o)e^{-\delta t_o} > 0$$

where $0 < x_o < \rho$, $t_o > 0$, and $\delta > 0$ is a constant. Let

$$v(x,t) \equiv w(x,t)e^{-\delta t} \quad \text{on} \quad [0,\rho] \times [0,\infty).$$

Then $w = ve^{\delta t}$ and substituting into the differential equation of (1.4) we have

$$d(v_x)_{xx}e^{\delta t} + f'(ve^{\delta t})v_x e^{\delta t} = (v_x e^{\delta t})_t = (v_x)_t e^{\delta t} + \delta v_x e^{\delta t}. \tag{1.5}$$

At (x_o,t_o), (1.5) implies

$$d(v_x)_{xx} = (v_x)_t + [\delta - f'(ve^{\delta t})]v_x \geq [\delta - f'(ve^{\delta t})]v_x > 0$$

and this contradicts the (weak) maximum principle. Thus a maximum of $w_x e^{-\delta t}$ must occur either at $x = 0$ or $x = \rho$ and so the same is true of w_x. This completes the proof of the lemma.

Since $w(\rho,t) = 0$ and if $\lim_{x \to \rho^-} w_x(x,t) = w_x(\rho^-,t)$ is positive, then w must be negative valued in a neighborhood of ρ. Using Lemma 1.1, we can show that w_x cannot be zero wherever w itself is zero.

Proposition 1.2. Suppose $w(x,t)$ is the solution to (1.4) and let $t = \tau$ be fixed but arbitrary. If $w(\bar{x},\tau) = 0$ where $\bar{x} < \rho$, then $w_x(\bar{x},\tau) < 0$.

Proof. Suppose (for contradiction) that $w(\bar{x},\tau) = w_x(\bar{x},\tau) = 0$ (for $\bar{x} < \rho$). By Lemma 1 of the appendix (which indicates some appropriate characteristics of Φ), there exists $\eta > 0$ such that $w_x(x,\tau) \leq 0$ for $x \in [\bar{x} - \eta, \bar{x} + \eta]$. Further, there exists $\delta > 0$ such that, for $(x,t) \in [\bar{x} - \eta, \bar{x} + \eta] \times [\tau - \delta, \tau]$, $w_x \leq 0$. If $w_x(x,\tau - \delta) = \chi(x)$ for $x \in [\bar{x} - \eta, \bar{x} + \eta]$, if $w_x(\bar{x} - \eta, t) = g_1(t) \leq 0$ and if $w_x(\bar{x} + \eta, t) = g_2(t) \leq 0$ for $\tau - \delta \leq t \leq \tau$, then consider the following problem:

$$\left.\begin{array}{lll} dz_{xx} + f'(w)z = z_t, & \bar{x} - \eta \leq x \leq \bar{x} + \eta, & \tau - \delta \leq t \leq \tau \\ z(\bar{x} - \eta, t) = g_1(t), & z(\bar{x} + \eta, t) = g_2(t), & t > 0 \\ z(x,\tau - \delta) = \chi(x) \leq 0, & \bar{x} - \eta < x < \bar{x} + \eta. & \end{array}\right\} \tag{1.6}$$

By uniqueness, the solution to (1.6) agrees with the x-partial of w in the region $[\bar{x} - \eta, \bar{x} + \eta] \times [\tau - \delta, \tau]$. Further, the (strong) maximum principle (see Theorem 2 of the appendix) requires that $z(\bar{x},\tau)$ be negative and we have a contradiction. Thus we conclude that $w_x(\bar{x},\tau) < 0$ and the proposition is proved.

Proposition 1.2 shows clearly that the function $u(x,t)$ in the solution pair to (1.3) does not agree with the solution of (1.4) at least in a neighborhood of the moving boundary $\gamma(t)$. Since the solution of (1.4) $w(x,t)$ eventually is negative valued (for time large enough), it is reasonable to guess that $u(x,t) \geq w(x,t)$ for $(x,t) \in [0,\rho] \times [0,\infty)$. To prove this inequality, we introduce the sequence $\{f_k\}_1^\infty$ of real-valued functions each of which satisfy the following properties:

$$\left.\begin{array}{ll}\text{(i)} & f_k(0) = 0 \\ \text{(ii)} & f_k \text{ is nonincreasing} \\ \text{(iii)} & \text{for } \xi > 0,\ f_k(\xi) \geq f(\xi) \quad \text{and} \quad \lim_{k\to\infty} f_k(\xi) = f(\xi).\end{array}\right\} \tag{1.7}$$

Next let $u^k(x,t)$ be the solution of

$$\left.\begin{array}{l} du_{xx}^{\ k} + f_k(u^k) = u_t^{\ k}, \quad 0 < x < \rho, \quad t > 0 \\ u_x^{\ k}(0,t) = \alpha u^k(0,t), \quad u^k(\rho,t) = 0, \quad t > 0 \\ u^k(x,0) = \Phi(x), \quad 0 < x < \rho \end{array}\right\} \tag{1.8}$$

Using a sequence $\{f_k\}_1^\infty$ satisfying the properties of (1.7), we will show in chapter 2 that the sequence $\{u^k(x,t)\}_1^\infty$ satisfying (1.8) converges to $u(x,t)$ from above. Thus, showing $u^k(x,t) \geq w(x,t)$ for each $k \geq 1$ will establish the final proposition of this chapter.

<u>Proposition 1.3</u>. Suppose that $u(x,t)$ is in the solution pair $\{u(x,t),\gamma(t)\}$ for (1.3) and that $w(x,t)$ is the solution of (1.4). Assuming that the initial value $\Phi(x)$ and the constant α are the same for both (1.3) and (1.4), then $w(x,t) \leq u(x,t)$.

<u>Proof</u>. Let $\delta > 0$ be constant. If u^k is the solution to (1.8), let $v(x,t) = u^k(x,t)e^{-\delta t}$. Also let $z(x,t) = w(x,t)e^{-\delta t}$. Then from the differential equation of (1.4) we have

$$d(ze^{\delta t})_{xx} + f(ze^{\delta t}) = (ze^{\delta t})_t, \quad 0 < x < \rho, \quad t > 0 \tag{1.9}$$

and from the differential equation in (1.8) we have

$$d(ve^{\delta t})_{xx} + f_k(ve^{\delta t}) = (ve^{\delta t})_t, \quad 0 < x < \rho, \quad t > 0. \tag{1.10}$$

Subtracting (1.10) from (1.9) yields

$$d(z - v)_{xx}e^{\delta t} + f(ze^{\delta t}) - f_k(ve^{\delta t}) = (z - v)_t e^{\delta t} + \delta(z - v)e^{\delta t}. \tag{1.11}$$

Suppose (for contradiction) that

$$\sup_{\substack{0\le x\le\rho \\ t>0}} \{z - v\} = z(x_o,t_o) - v(x_o,t_o) > 0 \quad \text{where} \quad 0 < x_o < \rho$$

From (1.11) we have at (x_o,t_o) that

$$\begin{aligned} d(z - v)_{xx} &= [f_k(ve^{\delta t_o}) - f(ze^{\delta t_o})]e^{-\delta t_o} + (z - v)_t + \delta(z - v) \\ &\ge [f_k(ve^{\delta t_o}) - f(ze^{\delta t_o})]e^{-\delta t_o} + \delta(z - v). \end{aligned}$$

By (iii) of (1.7), $f_k(ve^{\delta t_o}) \ge f(ve^{\delta t_o})$ and so at (x_o,t_o),

$$d(z - v)_{xx} \ge [f(ve^{\delta t_o}) - f(ze^{\delta t_o})]e^{-\delta t_o} + \delta(z - v).$$

By hypothesis $z(x_o,t_o) - v(x_o,t_o) > 0$ which implies

$$f(v(x_o,t_o)e^{\delta t_o}) \ge f(z(x_o,t_o)e^{\delta t_o})$$

and therefore

$$d(z - v)_{xx}(x_o,t_o) \ge \delta z(x_o,t_o) - \delta v(x_o,t_o) > 0.$$

This obviously contradicts the maximum principle and we know that a positive maximum for $z - v$ does not occur in $(0,\rho) \times (0,\infty)$. Also $z(\rho,t) - v(\rho,t) = 0$ for $t > 0$ from the boundary conditions. Further, if $z(0,t_o) > v(0,t_o)$ then

$$z_x(0,t_o) = \alpha z(0,t_o) > \alpha v(0,t_o) = v_x(0,t_o)$$

which implies (for $0 < \alpha < \infty$) that a maximum cannot occur at $x = 0$. If $\alpha = \infty$, $z(0,t_o) = v(0,t_o) = c_o$ and so $x_o \neq 0$ in that case. Finally, if $\alpha = 0$ then $z_x(0,t) - v_x(0,t) = 0$ and by Theorem 3 of the appendix, a maximum for $z - v$ cannot occur at $x = 0$. Thus we conclude that $z - v \leq 0$ on $[0,\rho] \times [0,\infty)$. Multiplication be $e^{\delta t}$ implies that

$$w(x,t) \leq u^k(x,t) \quad \text{for} \quad (x,t) \in [0,\rho] \times [0,\infty). \tag{1.12}$$

Assuming $u^k(x,t)$ converges to $u(x,t)$ from above then (1.12) makes the conclusion of the proposition true.

2. EXISTENCE AND BEHAVIOR OF SOLUTIONS

In this chapter, we consider existence of a solution pair $\{u(x,t),\gamma(t)\}$ to the moving boundary problem:

$$\left.\begin{array}{lll} du_{xx} + f(u) = u_t, & 0 < x < \gamma(t), & t > 0 \\ u_x(0,t) = \alpha u(0,t), & u(\gamma(t),t) = u_x(\gamma(t),t) = 0, & t > 0, \\ u(x,0) = \Phi(x), & 0 < x < \rho & \end{array}\right\} \qquad (2.1)$$

In (2.1) we continue the convention that when $\alpha = \infty$ the boundary condition at $x = 0$ is Dirichlet with $u(0,t) = c_o > 0$. Also let b_o be a positive constant and suppose $\phi \in \mathcal{D}$ where

$$\mathcal{D} = \{\phi \in L^2 : 0 \leq \phi(x) \leq b_o \text{ a.e.}\}.$$

Let ρ be a positive number and let L^2 denote the Hilbert space $L^2([0,\rho]); \mathbb{R})$ of all square summable functions on $[0,\rho]$ with

$$|\phi|_2 = \left[\int_0^\rho |\phi(x)|^2 dx\right]^{\frac{1}{2}} \quad \text{for all } \phi \in L^2.$$

Further let L^2_+ (the positive cone) be the set

$$L^2_+ = \{\phi \in L^2 : \phi(x) \geq 0\}.$$

Note that $\mathcal{D}$ satisfies $\mathcal{D} \subset L^2_+ \subset L^2$. Finally, we define

$$\hat{\mathcal{D}} = \{\phi \in \mathcal{D} : \text{there exists a real number } r = r(\phi) \text{ such that } \phi \text{ is nondecreasing on } [0,r] \text{ and nonincreasing on } [r,\rho]\}.$$

We can now state our existence result for (2.1).

Theorem 2.1. For each $\Phi \in \hat{\mathcal{D}}$ there exists a unique pair $\{u(x,t;\ \Phi),\gamma(t)\}$ satisfying (2.1) such that $u(x,t;\ \Phi)$ is defined on $[0,\rho] \times [0,\infty)$ and satisfies $u(\cdot,t;\ \phi) \in \hat{\mathcal{D}}$ for all $t \geq 0$. Additionally, if $S^\alpha(t)\phi \equiv u(\cdot,t;\ \phi)$ for $t \geq 0$, $\phi \in \hat{\mathcal{D}}$, then S^α is a (C_o) semigroup of nonlinear operators on $\hat{\mathcal{D}}$ and there exists a $k_\alpha > 0$ such that

(Q1) $S^\alpha(t) : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ and $S^\alpha(0)\phi = \phi$ for all $t \geq 0$, $\phi \in \hat{\mathcal{D}}$;

(Q2) $S^\alpha(t + s)\phi = S^\alpha(t)S^\alpha(s)\phi$ for all $t,s \geq 0$, $\phi \in \hat{\mathcal{D}}$;

(Q3) $|S^\alpha(t)\phi_1 - S^\alpha(t)\phi_2|_2 \leq |\phi_1 - \phi_2|_2 e^{-k_\alpha t}$ for all $t \geq 0$ and $\phi_1,\phi_2 \in \hat{\mathcal{D}}$; and

(Q4) $S^\alpha(t)\phi_1 \geq S^\alpha(t)\phi_2$ if $t \geq 0$ and $\phi_1,\phi_2 \in \hat{\mathcal{D}}$ with $\phi_1 \geq \phi_2$

Remark. The values for k_α are $k_o = \pi^2 d/4\rho^2$, $k_\infty = \pi^2 d/\rho^2$, and otherwise k_α lies in the open interval $(\pi^2 d/4\rho^2,\pi^2 d/\rho^2)$ depending uniquely on α. As further clarification for k_α in (Q3) where α is fixed and satisfies $0 < \alpha < \infty$, consider positive solutions of the following equation:

$$\sin(\lambda\rho) + \frac{\lambda}{\alpha}\cos(\lambda\rho) = 0 \tag{2.2}$$

We are interested in the smallest positive value of λ satisfying (2.2) since it is the smallest eigenvalue from Sturm-Liouville theory. Letting $h(\lambda)$ be the left side of (2.2), then we observe that $h(0) = 0$ and that h is strictly positive on $[\pi/2\rho,\pi/\rho]$. Also $h(\lambda)$ is nonincreasing and nonconstant on $[\pi/2\rho,\pi/\rho]$ and $h(\pi/\rho) = -\pi/\alpha\rho < 0$. So we know h has exactly one zero $\hat{\lambda} = \hat{\lambda}(\alpha)$ in the interval $(\pi/2\rho,\pi/\rho)$. For each α such that $0 < \alpha < \infty$, we let $k_\alpha = (\hat{\lambda}(\alpha))^2 d$.

We also remark that the solution $u(x,t;\ \Phi)$ guaranteed by Theorem 2.1 is a solution in the Hilbert space L^2 (see Theorem 3.1 of [1]) and has the following regularity property: if (x_o,t_o) satisfies $t_o > 0$ and

$0 < x_o < \gamma(t_o)$, then $u_t(x,t;\phi)$ and $u_{xx}(x,t;\phi)$ exist and are Hölder continuous in a neighborhood of (x_o,t_o) [and hence satisfy the differential equation in (2.1)].

The proof of Theorem 2.1 requires several preliminary results. We begin by defining the linear operator L on L^2 (for $d > 0$ constant) by

$$\left.\begin{aligned} &L\phi = d\phi'' \quad \text{for all } \phi \in \mathrm{Dom}(L) \text{ where} \\ &\mathrm{Dom}(L) = \{\phi \in L^2 : \phi \text{ and } \phi' \text{ are absolutely continuous} \\ &\qquad \text{and } \phi'' \in L^2\} \end{aligned}\right\} \tag{2.3}$$

Next we define the linear operator L^α on L^2 by restricting Dom(L) as follows:

$$\left.\begin{aligned} &L^\alpha\phi = d\phi'' \quad \text{for all } \phi \in \mathrm{Dom}(L^\alpha) \text{ where} \\ &\mathrm{Dom}(L^\infty) = \{\phi \in \mathrm{Dom}(L) : \phi(\rho) = 0 \text{ and } \phi(0) = c_o\} \text{ and} \\ &\mathrm{Dom}(L^\alpha) = \{\phi \in \mathrm{Dom}(L) : \phi(\rho) = 0 \text{ and } \phi'(0) = \alpha\phi(0)\} \text{ for } \alpha < \infty. \end{aligned}\right\} \tag{2.4}$$

Lemma 2.2. For all α such that $0 \le \alpha \le \infty$, the operator L^α is closed, is densely defined on L^2, and is the infinitesimal generator of a (C_o) semigroup $T^\alpha = \{T^\alpha(t) : t \ge 0\}$ on L^2. The semigroup T^α satisfies the following properties:

(i) $t \to T^\alpha(t)\Phi$ is continuous on $[0,\infty)$ for each $\Phi \in L^2$,

(ii) $T^\alpha(0) = I$ and $T^\alpha(t+s) = T^\alpha(t)T^\alpha(s)$ for $t,s \ge 0$,

(iii) $|T^\alpha(t)\Phi|_2 \le |\Phi|_2 e^{-k_\alpha t}$ for all $t \ge 0$, $\Phi \in L^2$,

where k_α has the value described in the remark after Theorem 2.1; and

(iv) $L^\alpha = \lim_{h\to 0^+} h^{-1}[T^\alpha(h)\Phi - \Phi]$ for all $\Phi \in \mathrm{Dom}(L^\alpha)$.

Proof. The lemma follows from standard results for elliptic and parabolic differential equations and self-adjoint operators in Hilbert space (see e.g. [6]). Note that $-k_\alpha$ is the largest eigenvalue of L^α.

Each of the semigroups T^α is analytic (see Proposition 6.4, p. 309 of [6]) and so, if $\Phi \in \mathcal{L}^2$ and if $w^\alpha(x,t) = [T^\alpha(t)\Phi](x)$ for $t \geq 0$, $x \in [0,\rho]$, then w^α is the (classical) solution to the linear parabolic problem

$$\begin{aligned} &d w_{xx}^{\alpha}(x,t) = w_t^{\alpha}(x,t, \quad t > 0,\ 0 < x < \rho, \\ &w_x^{\alpha}(0,t) = \alpha w^{\alpha}(0,t), \quad w^{\alpha}(\rho,t) = 0, \quad t > 0, \\ &w^{\alpha}(x,0) = \phi(x), \quad 0 < x < \rho. \end{aligned} \tag{2.5}$$

[In (2.5), we use the same convention for w^∞ as given after (2.1)]. Using the maximum principle it is easy to show that $T^\alpha(t) : \mathcal{L}^2_+ \to \mathcal{L}^2_+$ for all $t \geq 0$, and that T^α is order preserving, i.e. $T^\alpha(t)\Phi_1 \geq T^\alpha(t)\Phi_2$ whenever $t \geq 0$, $\Phi_1 \geq \Phi_2$, and $0 \leq \alpha \leq \infty$. Indeed, we have one further result of the maximum principle in which the semigroup T^∞ already includes the nonhomogeneous boundary condition $w^\infty(0,t) = c_o$.

Lemma 2.3. Define $\phi_o \in \mathcal{L}^2$ by $\phi_o(x) = c_o(\rho - x)/\rho$ for all $x \in [0,\rho]$. Then $T^\infty(t)(\Phi - \phi_o) + \phi_o \in \mathcal{D}$ and $T^\alpha(t)\Phi \in \mathcal{D}$ $(0 \leq \alpha < \infty)$ for all $t \geq 0$ and $\Phi \in \mathcal{D}$ (of course, we assume $c_o \leq b_o$).

Proof. Since $0 \leq \Phi(x) \leq b_o$ a.e. on $[0,\rho]$, the maximum principle confirms immediately that $0 \leq [T^\alpha(t)\Phi](x) \leq b_o$ $(0 \leq \alpha < \infty)$ for $t \geq 0$ and $\Phi \in \mathcal{D}$. With only slight modification in the argument, the maximum principle also confirms the lemma for $\alpha = \infty$.

Next we need to consider the nonlinear part of the differential equation in (2.1). We do so by creating approximating functions f_k such that $f_k(0) = 0$. Additionally, we suppose that $f_k : [0,\infty) \to \mathbb{R}$, $k = 1,2,\ldots$ satisfies the following:

$$
\begin{array}{rl}
\text{(i)} & \text{there exists } M_k > 0 \text{ such that } |f_k(\xi) - f_k(\eta)| \le M_k|\xi - \eta| \\
& \text{for all } \xi,\eta \in [0,b_o] \text{ and } k = 1,2,\ldots; \\
\text{(ii)} & f_k(0) = 0 \text{ and } f_k \text{ is nonincreasing on } [0,\infty) \text{ for } k = 1,2,\ldots; \\
\text{(iii)} & \text{for each } \varepsilon > 0 \text{ there is an } M^\varepsilon > 0 \text{ such that} \\
& |f_k(\xi) - f_k(\eta)| \le M^\varepsilon|\xi - \eta| \text{ for all} \\
& \xi,\eta \in [\varepsilon,b_o] \text{ and } k = 1,2,\ldots; \\
\text{(iv)} & f_k(\xi) \le f_n(\xi) \text{ if } \xi \in (0,b_o] \text{ and } k \ge n; \text{ and} \\
\text{(v)} & f(\xi) = \lim\limits_{k\to\infty} f_k(\xi) \text{ for all } \xi > 0.
\end{array}
\tag{2.6}
$$

Two examples of sequences $\{f_k\}_1^\infty$ satisfying (2.6) are

$$
f_k(\xi) = \begin{cases} k\xi f(1/k) & \text{if } 0 \le \xi < 1/k \\ f(\xi) & \text{if } 1/k \le \xi \end{cases} \qquad k = 1,2,\ldots \tag{2.7}
$$

and

$$
f_k(\xi) = \frac{\xi f(\xi)}{k^{-1} + \xi}, \quad \xi \ge 0, \quad k = 1,2,\ldots \tag{2.8}
$$

Note that the form (2.8) is consistent with the Michaelis-Menten model for enzyme kinetics and that the form (2.8) is continuously differentiable on $[0,\infty)$.

A third interesting example occurs if the function f is extended so that it is multivalued and that $-f$ is a subset of a maximal monotone graph. Then we define the subset $\overline{f}$ of $\mathbb{R} \times \mathbb{R}$ by

$$(\xi,\eta) \in \bar{f} \iff \xi \geq 0 \text{ and } \begin{cases} \eta = f(\xi) & \text{if } \xi > 0 \\ \eta \geq f(0) & \text{if } \xi = 0 \end{cases}$$

In other words, $\bar{f}(\xi) = \{f(\xi)\}$ if $\xi > 0$ and $\bar{f}(0) = [f(0),\infty)$. In this case, $(I - \frac{1}{k}\bar{f})^{-1}$ is a nonexpansive function on $\mathbb{R}$ and we can use

$$f_k(\xi) = k[(I - k^{-1}\bar{f})^{-1}(\xi) - \xi] \quad \text{for } \xi \geq 0, \quad k = 1,2,... \tag{2.9}$$

(see, e.g., Brezis [1]). Of course there are many other possibilities for the sequence $\{f_k\}_1^\infty$ which will satisfy (2.6).

Given a selection for the approximating sequence $\{f_k\}_1^\infty$, we define the sequence $\{F_k\}_1^\infty$ of substitution operators on $\mathcal{D}$ by

$$[F_k\phi](x) = f_k(\phi(x)) \text{ for } x \in [0,\rho],\ \phi \in \mathcal{D},\ k = 1,2,... . \tag{2.10}$$

Since each f_k is Lipschitz on $[0,b_o]$, so is F_k and F_k maps $\mathcal{D}$ into L^2. Other properties of the sequence $\{F_k\}_1^\infty$ are given in the following:

Lemma 2.4. Suppose that f_k satisfies (2.6) and that F_k is defined by (2.10) for each $k \geq 1$. Then

(i) $F_k : \mathcal{D} \to L^2$ and $|F_k\phi_1 - F_k\phi_2|_2 \leq M_k|\phi_1 - \phi_2|_2$ for all $\phi_1,\phi_2 \in \mathcal{D}$ and $k = 1,2,...;$

(ii) $|\phi_1 - \phi_2 - h[F_k\phi_1 - F_k\phi_2]|_2 \geq |\phi_1 - \phi_2|_2$ for all $\phi_1,\phi_2 \in \mathcal{D}$, $h > 0$ and $k = 1,2,...;$ and

(iii) $\lim_{h\to 0^+} d_2(\phi + hF_k\phi; \mathcal{D})/h = 0$ for all $\phi \in \mathcal{D}$ and $k = 1,2,...$, where $d_2(\chi; \mathcal{D}) \equiv \inf\{|\chi - \psi|_2 : \psi \in \mathcal{D}\}$ for each $\chi \in L^2$.

Proof. (i) is an immediate result of (2.6) (i). Next since each f_k is nonincreasing, if $\phi_1(x) \geq \phi_2(x)$, then $f_k(\phi_1(x)) \leq f_k(\phi_2(x))$. This implies

$$|\phi_1(x) - \phi_2(x) - h[f_k(\phi_1(x)) - f_k(\phi_2(x))]| = |\phi_1(x) - \phi_2(x) + hf_k(\phi_2(x)) - hf_k(\phi_1(x))| = \phi_1(x) - \phi_2(x) + h[f_k(\phi_2(x)) - f_k(\phi_1(x))] \geq \phi_1(x) - \phi_2(x) = |\phi_1(x) - \phi_2(x)|.$$

Similarly, if $\phi_1(x) < \phi_2(x)$, then

$$|\phi_1(x) - \phi_2(x) - h[f_k(\phi_1(x)) - f_k(\phi_2(x))]| \geq |\phi_1(x) - \phi_2(x)|$$

Thus, for all $x \in [0,\rho]$ we have

$$|\phi_1(x) - \phi_2(x) - h[f_k(\phi_1(x)) - f_k(\phi_2(x))]| \geq |\phi_1(x) - \phi_2(x)|. \quad (2.11)$$

Now then (ii) follows by squaring each side of (2.11) and integrating from $x = 0$ to $x = \rho$. Finally, since $f_k(0) = 0$ and $f_k(b_o) \leq 0$, for each $\varepsilon > 0$ there is a continuously differentiable function $g_{k,\varepsilon} : [0,b_o] \to \mathbb{R}$ such that

$$|g_{k,\varepsilon}(\xi) - f_k(\xi)| \leq \varepsilon \quad \text{for } \xi \in [0,b_o]$$

and such that $g_{k,\varepsilon}(0) > 0$ and $g_{k,\varepsilon}(b_o) < 0$. Now we define the operator $G_{k,\varepsilon} : \mathcal{D} \to \mathcal{L}^2$ by

$$[G_{k,\varepsilon}\phi](x) = g_{k,\varepsilon}(\phi(x)) \quad \text{for all } \phi \in \mathcal{D} \text{ and } x \in [0,\rho].$$

Then, whenever $\phi \in \mathcal{D}$ and $h > 0$ is sufficiently small, $\phi + hG_{k,\varepsilon}\phi$ is in $\mathcal{D}$ and therefore,

$$\lim_{h\to 0^+} d_2(\phi + hG_{k,\varepsilon}\phi\,;\,\mathcal{D})/h = 0 \quad \text{for all } \phi \in \mathcal{D}.$$

Since $|F_k\phi - G_{k,\varepsilon}\phi|_2^2 \leq \varepsilon^2\rho$ for all $\varepsilon > 0$, we conclude that (iii) is also valid. This completes the proof of the lemma.

Using Lemmas 2.3 and 2.4, we can apply Theorem 3.2 (chapter 8) of [6] to show that for each $k \geq 1$ the L^2-valued integral equations

$$S_k^{\infty}(t)\Phi = T^{\infty}(t)(\Phi - \phi_o) + \phi_o + \int_0^t T^{\infty}(t - s)F_k S_k^{\infty}(s)\Phi ds \tag{2.12}$$

(with $\phi_o(x) = c_o(\rho - x)/\rho$ for $x \in [0,\rho]$) and

$$S_k^{\alpha}(t)\Phi = T^{\alpha}(t)\Phi + \int_0^t T^{\alpha}(t - s)F_k S_k^{\alpha}(s)\Phi ds, \quad 0 \leq \alpha < \infty \tag{2.13}$$

have solutions for all $\Phi \in \mathcal{D}$. Also $S_k^{\alpha} : \mathcal{D} \to \mathcal{D}$ and we state other properties for the families $S_k^{\alpha} = \{S_k^{\alpha}(t);\ t \geq 0\}$ in the following lemma.

<u>Lemma 2.5</u>. The operator families $S_k^{\alpha} = \{S_k^{\alpha}(t);\ t \geq 0\}$ from (2.12) and (2.13) satisfy the following properties:

(P1) $S_k^{\alpha}(0)\Phi = \Phi$ and $S_k^{\alpha}(t)S_k^{\alpha}(s)\Phi = S_k^{\alpha}(t + s)\Phi$ for $\Phi \in \mathcal{D}$ and $t,s \geq 0$;

(P2) $|S_k^{\alpha}(t)\Phi_1 - S_k^{\alpha}(t)\Phi_2|_2 \leq e^{-k_\alpha t}|\Phi_1 - \Phi_2|_2$ for any $\Phi_1,\Phi_2 \in \mathcal{D}$, $t \geq 0$;

(P3) $S_k^{\alpha}(t)\Phi_1 \geq S_k^{\alpha}(t)\Phi_2$ for $t \geq 0$, $\Phi_1,\Phi_2 \in \mathcal{D}$ where $\Phi_1 \geq \Phi_2$; and

(P4) $S_k^{\alpha}(t)\Phi \leq S_n^{\alpha}(t)\Phi$ for $t \geq 0$, $\Phi \in \mathcal{D}$ and $k \geq n$.

<u>Proof</u>. (P1) results from the existence of a unique solution to (2.12) and (2.13). (P2) follows from the dissipative property of F_k [see (ii) of Lemma 2.4] along with property (iii) of T^{α} given in Lemma 2.2. (P3) and (P4) follow from the maximum principle and the fact that $f_k(\xi) \leq f_n(\xi)$ whenever $k \geq n$ and $0 \leq \xi \leq b_o$. For since T^{α} is analytic and f_k is Lipschitz continuous on $[0,b_o]$, regularity results in

Friedman (see p. 204 of [3]) show that if $\Phi \in \mathcal{D}$ and if $w_k^{\alpha}(x,t) \equiv [S_k^{\alpha}(t)\Phi](x)$ for $t \geq 0$, $0 \leq x \leq \rho$, then w_k^{α} satisfies

$$\left.\begin{aligned} &\partial_t w_k^{\alpha}(x,t) = d\partial_{xx} w_k^{\alpha}(x,t) + f_k(w_k^{\alpha}(x,t)),\ t > 0, \\ &\quad 0 < x < \rho. \\ &\partial_x w_k^{\alpha}(0,t) = \alpha w_k^{\alpha}(0,t),\ w_k^{\alpha}(\rho,t) = 0,\ t \geq 0, \\ &w_k^{\alpha}(x,0) = \Phi(x),\ 0 < x < \rho. \end{aligned}\right\} k = 1,2,\ldots \qquad (2.14)$$

This completes the proof of the lemma.

Using (P4) of Lemma 2.5 and the fact that members of $\mathcal{D}$ are non-negative, we have $S_n^{\alpha}(t)\Phi \geq S_k^{\alpha}(t)\Phi \geq \theta$ (the zero function) for all $t \geq 0$, $\Phi \in \mathcal{D}$ and $k \geq n$. Thus $\lim_{k\to\infty} S_k^{\alpha}(t)\Phi$ exists for all $t \geq 0$ and for $\Phi \in \mathcal{D}$ because of monotone convergence. Indeed, we will show in chapter 3 that the semigroups $\{S_k^{\alpha}\}_1^{\infty}$ converge (uniformly) to the semigroup S^{α} generated by the solution to problem (2.1). It is necessary now to state regularity properties of the semigroup S^{α}. Suppose that $\Phi \in \mathcal{D}$ and let

$$u(x,t) \equiv [S^{\alpha}(t)\Phi](x) \quad \text{for all } t \geq 0,\quad x \in [0,\rho] \qquad (2.15)$$

where

$$[S^{\alpha}(t)\Phi](x) = \lim_{k\to\infty} [S_k^{\alpha}(t)\Phi](x). \qquad (2.16)$$

Proposition 2.6. Suppose $\Phi \in \mathcal{D}$ and $u(x,t)$ is defined by (2.15). Then the limit in (2.16) is uniform for each $0 < \delta < R$ such that $(x,t) \in [0,\rho] \times [\delta,R]$. Also S^{α} satisfies properties Q1 - Q4 in Theorem 2.1. Further, $u_x(x,t)$ exists for all $(x,t) \in [0,\rho] \times (0,\infty)$ and there is a constant $Q > 0$ (independent of $\Phi \in \mathcal{D}$) such that

$$\left.\begin{array}{l} |u_x(x,t) - u_x(y,t)| \le Q(1 + \frac{1}{t})|x - y|^{\frac{1}{2}} \\ \quad \text{for all } t > 0 \text{ and } x,y \in [0,\rho]. \end{array}\right\} \tag{2.17}$$

Moreover, for each $0 < \delta < R$ there are constants $P = P(\delta,R) > 0$ and $\nu = \nu(\delta,R) \in (0,1)$ such that

$$\left.\begin{array}{l} |u(x,t) - u(y,s)| \le P(|x - y|^{\nu} + |t - s|^{\nu}) \\ \quad \text{for all } 0 \le x,y \le \rho \text{ and } \delta \le t,s \le R. \end{array}\right\} \tag{2.18}$$

Proof. The proof of this proposition will be given in chapter 3 because it involves maximal monotone operator theory.

Before stating the next proposition, we need to recall a proper subset of $\mathcal{D}$, namely

$$\begin{array}{l} \hat{\mathcal{D}} = \{\phi \in L^2 : 0 \le \phi(x) \le b_o \text{ for } x \in [0,\rho] \text{ and there} \\ \quad \text{exists a real number } r = r(\phi) \text{ in } [0,\rho] \text{ such that} \\ \quad \phi \text{ is nondecreasing on } [0,r] \text{ and nonincreasing on} \\ \quad [r,\rho]\}. \end{array} \tag{2.19}$$

Observe that $\hat{\mathcal{D}}$ is a closed, bounded subset of L^2 and includes nonincreasing ($r = 0$) and nondecreasing ($r = \rho$) functions on $[0,\rho]$. The last result needed to prove Theorem 2.1 is a proposition.

Proposition 2.7. Suppose that S^{α} is defined on $[0,\infty) \times \mathcal{D}$ by (2.16), that $\Phi \in \hat{\mathcal{D}}$, and that $u(x,t) \equiv [S^{\alpha}(t)\Phi](x)$ for all $(x,t) \in [0,\rho] \times [0,\infty)$. Also, for each $t > 0$ define $\gamma(t) = \sup\{x \in [0,\rho] : u(x,t) > 0\}$. Then $\{u(x,t),\gamma(t)\}$ is the solution to (2.1) with initial value Φ.

<u>Proof</u>. Given α fixed such that $0 \le \alpha \le \infty$ and for each $k \ge 1$, we define $u^k(x,t) = [S_k^{\alpha}(t)\Phi](x)$ for all $(x,t) \in [0,\rho] \times [0,\infty)$. From (2.16) $u(x,t) = \lim_{k\to\infty} u^k(x,t)$ for all $(x,t) \in [0,\rho] \times [0,\infty)$ and this limit is uniform for $0 < \delta \le t \le R$ and $0 \le x \le \rho$. Moreover, as will be shown in chapter 3,

$$\partial_x u(x,t) = u_x(x,t) = \lim_{k\to\infty} \partial_x u^k(x,t) = \lim_{k\to\infty} u_x^k(x,t)$$

$$\text{for } (x,t) \in [0,\rho] \times (0,\infty)$$

and this limit is uniform for $0 < \delta \le t \le R$ and $0 \le x \le \rho$. Thus, u is continuous on $[0,\rho] \times (0,\infty)$ and u_x is continuous on $[0,\rho]$ for each $t > 0$.

Since for each $k \ge 1$, $u^k(x,t)$ satisfies the boundary conditions of (2.14), we know that $u_x^k(0,t) = \alpha u^k(0,t)$ and $u^k(\rho,t) = 0$ for $t > 0$. Hence, $u_x(0,t) = \alpha u(0,t)$ and $u(\rho,t) = 0$ for $t > 0$. Also, if $\Phi \in \hat{\mathcal{D}}$, we will show later (see Proposition 3.8) that for each $t > 0$, $S^{\alpha}(t) : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$. Hence $u(\cdot,t;\ \Phi) \in \hat{\mathcal{D}}$ for each $t \ge 0$ and $\Phi \in \hat{\mathcal{D}}$. Further, if $\gamma(t) = \rho$ then $u(x,t) > 0$ for $0 \le x < \rho$ and $u(\rho,t) = 0$; and if $\gamma(t) < \rho$, then $u(x,t) > 0$ for $0 \le x < \gamma(t)$ and $u(x,t) = 0$ for $\gamma(t) \le x \le \rho$. Since $u_x(\cdot,t)$ is continuous and $u(x,t) \equiv 0$ on $[\gamma(t),\rho]$, we have also that $u_x(\gamma(t),t) = 0$ whenever $\gamma(t) < \rho$. Thus, $\{u(x,t),\gamma(t)\}$ satisfies the initial and boundary conditions in (2.1). What remains is to show that the differential equation in (2.1) is satisfied for $t > 0$ and $0 < x < \gamma(t)$.

Suppose that $t_o > 0$ and $0 < x_o < \gamma(t_o)$ are given. Then $u(x_o,t_o) > 0$ and (by the continuity of u) there exists $\delta > 0$ such that $\delta < t_o$, $[x_o - \delta, x_o + \delta] \subset (0,\rho)$, and such that $u(x,t) \ge \varepsilon > 0$ for all (x,t) in $[x_o - \delta, x_o + \delta] \times [t_o - \delta, t_o + \delta]$. Since the functions f_k converge

monotonically to f and f is continuous on $[\varepsilon,b_o]$, it follows that $f_k \to f$ uniformly on $[\varepsilon,b_o]$ as $k \to \infty$. Hence $f(u(x,t)) = \lim_{k\to\infty} f_k(u^k(x,t))$ uniformly for $(x,t) \in [x_o - \delta, x_o + \delta] \times [t_o - \delta, t_o + \delta]$. By (iii) of (2.6) there exists an $M^\varepsilon > 0$ such that

$$\left.\begin{array}{l} |f_k(u^k(x,t)) - f_k(u^k(y,s))| \leq M^\varepsilon |u^k(x,t) - u^k(y,s)| \text{ where} \\ (x,t),(y,s) \in [x_o - \delta, x_o + \delta] \times [t_o - \delta, t_o + \delta] \text{ and } k = 1,2,\ldots \end{array}\right\} \quad (2.20)$$

Now (2.20) holds with f_k replaced by f and u^k replaced by u. Then we have by using (2.18) and by the fact that $\{S_k^\alpha(t)\phi\}_{k=1}^\infty$ is equicontinuous with Hölder coefficient independent of k that for $0 < \nu < 1$ there is a constant $K_\nu > 0$ such that

$$|f_k(u^k(x,t)) - f_k(u^k(y,s))| \leq K_\nu(|t - s|^\nu + |x - y|^\nu)$$

and

$$|f(u(x,t)) - f(u(y,s))| \leq K_\nu(|t - s|^\nu + |x - y|^\nu)$$

whenver $(x,t),(y,s) \in [x_o - \delta, x_o + \delta] \times [t_o - \delta, t_o + \delta]$ and $k = 1,2,\ldots$. Since K_ν is independent of all integers k, we can apply a result by Friedman (see Theorem 15 (p. 80) of [3]) to conclude that u(x,t) satisfies $du_{xx} + f(u) = u_t$ for all $(x,t) \in [x_o - \delta, x_o + \delta] \times [t_o - \delta, t_o + \delta]$. This completes the proof of Proposition 2.7.

The conclusions of Propositions 2.6 and 2.7 then verify Theorem 2.1. Note that in Proposition 2.7, Φ was required to be in $\hat{\mathcal{D}}$. While $S^\alpha(t)\phi$ exists for all $t \geq 0$ and $\phi \in \mathcal{D}$ ($S^\alpha(t) : \mathcal{D} \to \mathcal{D}$), there remain unanswered questions concerning whether the solution is classical for x satisfying $0 < x < \gamma(t)$, whenever $\phi \in \mathcal{D}$ but $\phi \notin \hat{\mathcal{D}}$.

3. PRODUCT FORMULA AND INVARIANT SETS

The major result of this chapter is that the set $\hat{\mathcal{D}}$ is invariant relative to the nonlinear semigroup $S^\alpha = \{S^\alpha(t) : t \geq 0\}$ defined in (2.16). Recall that

$$\mathcal{D} = \{\phi \in L^2 : 0 \leq \phi(x) \leq b_o \text{ a.e.}\}.$$

Now we define a subset $\mathcal{D}^\alpha$ of $\mathcal{D}$ as

$$\begin{aligned} \mathcal{D}^\alpha \equiv \{\phi \in \mathcal{D} : \phi \text{ and } \phi' \text{ are absolutely continuous, } \phi'' \in L^2, \\ \phi'(0) = \alpha\phi(0) \text{ and } \phi(\rho) = 0\} \end{aligned} \tag{3.1}$$

Also we define the multivalued operator F on $\mathcal{D}$ by

$$\left.\begin{aligned} &(\phi,\zeta) \in \text{Graph (F) only if } \phi \in \mathcal{D}, \text{ if} \\ &\zeta(x) = f(\phi(x)) \text{ for } x \in [0,\rho] \text{ such that } \phi(x) > 0, \text{ and if} \\ &f(0) \leq \zeta(x) \leq 0 \text{ for } x \in [0,\rho] \text{ such that } \phi(x) = 0. \end{aligned}\right\} \tag{3.2}$$

Last we define the multivalued operator A^α by

$$A^\alpha\phi = \{d\phi'' + F\phi\} \text{ for all } \phi \in \mathcal{D}^\alpha. \tag{3.3}$$

With these definitions we may begin by proving a lemma.

<u>Lemma 3.1</u>. Suppose for $0 \leq \alpha \leq \infty$ that A^α is defined by (3.3). Then where it is defined, $(I - hA^\alpha)^{-1}$ satisfies (for $h > 0$)

$$|(I - hA^\alpha)^{-1}\phi - (I - hA^\alpha)^{-1}\bar{\phi}|_2 \leq (1 + hk_\alpha)^{-1}|\phi - \bar{\phi}|_2 \tag{3.4}$$

for all $\phi,\bar{\phi}$ in the range $R(I - hA^\alpha)$ and k_α as defined in Theorem 2.1.

Proof. If (ϕ,ζ), $(\overline{\phi},\overline{\zeta}) \in A^\alpha$ then clearly $\phi - \overline{\phi}$ is in $\mathrm{Dom}(L^\alpha)$ and if ζ_o and $\overline{\zeta}_o$ are such that

$$\zeta(x) = d\phi''(x) + \zeta_o(x) \quad \text{and} \quad \overline{\zeta}(x) = d\overline{\phi}''(x) + \overline{\zeta}_o(x)$$

then after combining we have

$$\zeta - \overline{\zeta} = d(\phi - \overline{\phi})'' + \zeta_o - \overline{\zeta}_o.$$

Since L^α is the generator of the semigroup T^α which satisfies (iii) of Lemma 2.2, we have

$$\int_0^\rho (\phi - \overline{\phi})(\zeta - \overline{\zeta})dx = \int_0^\rho (\phi - \overline{\phi})d(\phi - \overline{\phi})''dx + \int_0^\rho (\phi - \overline{\phi})(\zeta_o - \overline{\zeta}_o)dx$$

where

$$\int_0^\rho (\phi - \overline{\phi})d(\phi - \overline{\phi})''dx = \int_0^\rho (\phi - \overline{\phi})L^\alpha(\phi - \overline{\phi})dx \leq -k_\alpha \int_0^\rho (\phi - \overline{\phi})^2dx.$$

Consider the following subsets of $[0,\rho]$:

$$N_1 = \{x \in [0,\rho] : \phi(x) > 0 \text{ and } \overline{\phi}(x) > 0\},$$

$$N_2 = \{x \in [0,\rho] : \phi(x) > 0 \text{ and } \overline{\phi}(x) = 0\}, \text{ and}$$

$$N_3 = \{x \in [0,\rho] : \phi(x) = 0 \text{ and } \overline{\phi}(x) > 0\}.$$

Then

$$\left.\begin{aligned}\int_0^\rho (\phi - \overline{\phi})(\zeta_o - \overline{\zeta}_o)dx &= \int_{N_1} [\phi(x) - \overline{\phi}(x)][f(\phi(x)) - f(\overline{\phi}(x))]dx \\ &+ \int_{N_2} \phi(x)[f(\phi(x)) - \overline{\zeta}_o(x)]dx + \int_{N_3} -\overline{\phi}(x)[\zeta_o(x) - f(\overline{\phi}(x))]dx.\end{aligned}\right\} \quad (3.5)$$

In (3.5), the integral over N_1 is nonpositive since f is nonincreasing; and the integral over N_2 is nonpositive since $f(0) \leq \bar{\zeta}_o(x) \leq 0$ which implies

$$f(\phi(x)) - \bar{\zeta}_o(x) \leq f(0) - \bar{\zeta}_o(x) \leq 0.$$

With similar argument, the integral over the set N_3 in (3.5) is also nonpositive and so we have that

$$\begin{aligned}\int_0^\rho (\phi - \bar{\phi})(\zeta - \bar{\zeta})dx &\leq \int_0^\rho (\phi - \bar{\phi})L^\alpha(\phi - \bar{\phi})dx \leq -k_\alpha \int_0^\rho (\phi - \bar{\phi})^2 dx \\ &= -k_\alpha|\phi - \bar{\phi}|_2^{\,2}\end{aligned} \tag{3.6}$$

for all (ϕ,ζ), $(\bar{\phi},\bar{\zeta}) \in A^\alpha$. Then using (3.6) we have

$$\begin{aligned}&[|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2 - |\phi - \bar{\phi}|_2]/(-h) \\ &\leq \lim_{h\to0^+} [|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2 - |\phi - \bar{\phi}|_2]/(-h) \\ &= \lim_{h\to0^+} \frac{|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2^{\,2} - |\phi - \bar{\phi}|_2^{\,2}}{(-h)[|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2 + |\phi - \bar{\phi}|_2]} \\ &= \lim_{h\to0^+} \frac{-2h\int_0^\rho (\phi - \bar{\phi})(\zeta - \bar{\zeta})dx}{(-h)[|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2 + |\phi - \bar{\phi}|_2]} \\ &\leq \lim_{h\to0^+} -2k_\alpha|\phi - \bar{\phi}|_2^{\,2}/[|(\phi - h\zeta) - (\bar{\phi} - h\zeta)|_2 + |\phi - \bar{\phi}|_2] \\ &= -2k_\alpha|\phi - \bar{\phi}|_2^{\,2}/(2|\phi - \bar{\phi}|_2) = -k_\alpha|\phi - \bar{\phi}|_2.\end{aligned} \tag{3.7}$$

Then (3.7) implies that

$$|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2 - |\phi - \bar{\phi}|_2 \geq hk_\alpha|\phi - \bar{\phi}|_2$$

or

$$|(\phi - h\zeta) - (\bar{\phi} - h\bar{\zeta})|_2 \geq (1 + hk_\alpha)|\phi - \bar{\phi}|_2$$

for all (ϕ,ζ), $(\bar{\phi},\bar{\zeta}) \in A^\alpha$ and $h > 0$. Therefore, (3.4) is valid for all $\phi,\bar{\phi}$ in the range $R(I - hA^\alpha)$ and the lemma is proved.

Lemma 3.2. For each $h > 0$, the range of $I - hA^\alpha$ contains $\mathcal{D}$.

Proof. Observe that the multivalued operator $-A^\alpha$ defined by (3.3) has a maximal monotone extension $-B^\alpha$ such that $R(I - hB^\alpha) = L^2 \supset \mathcal{D}$ (see Theorem 5 (p. 42) of [4]). The lemma then follows from the maximum principle in a manner similar to the proof of Corollary 3.10.

Before establishing the product formula, we need to show that the semigroup generated by A^α is the limit of the sequence of semigroups $\{S_k^\alpha\}_{k=1}^\infty$ (see Lemma 2.5). We define the operator A_k^α on its domain $\mathcal{D}^\alpha$ as

$$A_k^\alpha\phi = d\phi'' + f_k(\phi), \tag{3.8}$$

and we discuss A_k^α in the following proposition.

Proposition 3.3. Suppose α satisfies $0 \leq \alpha \leq \infty$, $\phi \in \mathcal{D}$ and $h > 0$. Then

$$T_k = (I - hA_k^\alpha)^{-1}\phi \in \mathcal{D} \text{ for all } k = 1,2,\dots , \tag{3.9}$$

$T_j \leq T_k$ if $j \geq k$, and

$$\lim_{k\to\infty} (I - hA_k^\alpha)^{-1}\phi = (I - hA^\alpha)^{-1}\phi. \tag{3.10}$$

Proof. Since $\mathcal{D}$ is closed and convex, the fact that $S_k^\alpha(\cdot) : \mathcal{D} \to \mathcal{D}$ implies that (3.9) is valid (see Proposition 5.3 (p. 357) of [6]). Also since $f_j(\xi) \leq f_k(\xi)$ when $j \geq k$ and $\xi \geq 0$, we have by the maximum principle that $T_j \leq T_k$ for $j \geq k$. Therefore the limit

$$T(x) = \lim_{k\to\infty} T_k(x) \quad \text{exists for all} \quad x \in [0,\rho] \tag{3.11}$$

and since

$$dT_k''(x) = \frac{T_k(x) - \Phi(x)}{h} - f_k(T_k(x)), \quad k = 1,2,\ldots \tag{3.12}$$

it follows that $\sup\{|T_k''(x)| : x \in [0,\rho], k \geq 1\} < \infty$. Then it follows easily that $\{T_k\}_1^\infty$ and $\{T_k'\}_1^\infty$ converge uniformly to T and T' respectively and that T and T' are absolutely continuous. Now $T \in \mathcal{D}$ because $\mathcal{D}$ is closed and let

$$N(T) = \{x \in [0,\rho] \text{ such that } T(x) = 0\}$$

and let $I(T)$ be the relative complement of $N(T)$, i.e.

$$I(T) = [0,\rho] - N(T).$$

By (v) of (2.6), if $x \in I(T)$ then

$$\lim_{k\to\infty} f_k(T_k(x)) = f(T(x)).$$

Thus T'' exists on $I(T)$ and

$$dT''(x) = \frac{T(x) - \Phi(x)}{h} - f(T(x)) \quad \text{for} \quad x \in I(T).$$

If

$$\chi(x) = \begin{cases} dT''(x) + f(T(x)) & \text{for} \quad x \in I(T) \\ -h^{-1}\Phi(x) & \text{for} \quad \in N(T) \end{cases}$$

then $T - h\chi = \Phi$. To complete the proof we need to show that $\chi \in A^\alpha T$ since obviously $T \in \mathcal{D}^\alpha$, the domain of A^α. It suffices to show that $\chi(x) \geq dT''(x) + f(0)$ for $x \in N(T)$ because then

$$\chi(x) \in [dT''(x) + f(0),\infty) \quad \text{for} \quad x \in N(T).$$

Suppose $x \in N(T)$ and y satisfies $x < y \leq \rho$. Then integrating each side of (3.12) from x to y yields

$$hdT_k'(y) - hdT_k'(x) = \int_x^y T_k(r)dr - \int_x^y \Phi(r)dr - h\int_x^y f_k(T_k(r))dr.$$

Since

$$-h\int_x^y f_k(T_k(r))dr \leq -h\int_x^y f(T_k(r))dr \to -hf(\xi(y))(y - x)$$

as $k \to \infty$ where $0 = T(x) \leq \xi(y) \leq T(y)$, we know that

$$hdT'(y) - hdT'(x) - \int_x^y T(r)dr + \int_x^y \Phi(r)dr \leq -hf(\xi(y))(y - x)$$

for any x in $N(T)$. Dividing by $(y - x)$, we have

$$hd\left[\frac{T'(y) - T'(x)}{y - x}\right] + hf(\xi(y)) \leq (y - x)^{-1}\int_x^y T(r)dr$$
$$- (y - x)^{-1}\int_x^y \Phi(r)dr,$$

and so, as $y \to x^+$, we have

$$h[dT''(x) + f(0)] \leq T(x) - \Phi(x) = -\Phi(x)$$

for almost all $x \in N(T)$. Thus

$$\chi(x) = -h^{-1}\Phi(x) \geq dT''(x) + f(0)$$

and the proposition is proved.

Lemma 3.4. Suppose $S^\alpha = \{S^\alpha(t) : t \geq 0\}$ and $S_k^\alpha = \{S_k^\alpha(t) : t \geq 0\}$ are the semigroups generated by operators A^α and A_k^α respectively. Then

$$S^{\alpha}(t)\Phi = \lim_{k\to\infty} S_k{}^{\alpha}(t)\Phi \quad \text{for all} \quad \Phi \in \mathcal{D} \tag{3.13}$$

and this limit is uniform on compact t-intervals.

Proof. Using (3.10), the conclusion is immediate from Brezis (see Proposition 4.2 (p. 122) of [1]).

With Lemmas 3.1, 3.2, and 3.4 and Proposition 3.3 available, we can state and prove the very useful product formula.

Theorem 3.5. Let A^{α} be defined by (3.3) and let $\Phi \in \mathcal{D}$. Then for each $t > 0$

$$S^{\alpha}(t)\Phi = \lim_{n\to\infty} (I - \frac{t}{n} A^{\alpha})^{-n}\Phi \tag{3.14}$$

and the limit is uniform on bounded t-invervals.

Proof. By Lemma 3.2, $R(I - hA^{\alpha}) \supset \mathcal{D}$ for all $h > 0$ and since (3.4) holds, we know that

$$\overline{S}^{\alpha}(t)\Phi \equiv \lim_{n\to\infty} (I - \frac{t}{n} A^{\alpha})^{-n}\Phi$$

exists for all $\Phi \in \mathcal{D}$, uniformly on bounded t-intervals (see, e.g., Theorem 4.3 (p. 124) of [1]). By (3.10) we have

$$\lim_{k\to\infty} (I - hA_k{}^{\alpha})^{-1}\Phi = (I - hA^{\alpha})^{-1}\Phi$$

for all $h > 0$ and all $\Phi \in \mathcal{D}$. Since $A_k{}^{\alpha}$ is the generator of $S_k{}^{\alpha}$ and (3.13) holds, it follows that $\overline{S}^{\alpha} = S^{\alpha}$ and that (3.14) is valid. This completes the proof of the theorem.

Before proceeding to a discussion of invariant sets, we will prove regularity results needed in chapter 2 for existence of a solution to (2.1). We begin by indicating that the generator $-A^{\alpha}$ [from (3.3)] is the subdifferential of a lower semicontinuous, convex function. Since the linear part of $-A^{\alpha}$ (that is, the second derivative operator along with the boundary conditions) is self-adjoint (modulo an inhomogeneous boundary term in the case $\alpha = \infty$), this term is a subdifferential according to Brezis (see Proposition 2.15 (p. 47) of [1]). Also, by Example 2.8.1 (p. 43) of [1] and by Proposition 2.16 (p. 47) of [1], we have that the multivalued part of $-A^{\alpha}$ generated by $-F$ is also a subdifferential. Then since $-A^{\alpha}$ has a maximal monotone extension, we see that $-A^{\alpha}$ is the subdifferential of a lower semicontinuous, proper, convex function on L^2. Therefore, by Theorem 3.2 (p. 57) of [1], $S^{\alpha}(t)\phi \in \mathcal{D}^{\alpha}$ for all $t > 0$, $\phi \in \mathcal{D}$, and the following inequality holds:

$$|\overset{\circ}{A}{}^{\alpha}S^{\alpha}(t)\phi|_2 \leq |\overset{\circ}{A}{}^{\alpha}T|_2 + \frac{1}{t}\,|\phi - T|_2 \quad \text{for all} \quad t > 0,$$

$$\phi \in \mathcal{D} \quad \text{and} \quad T \in \mathcal{D}^{\alpha}$$

where $\overset{\circ}{A}{}^{\alpha}$ is the minimum section of A^{α} and $|\overset{\circ}{A}{}^{\alpha}T|_2 \equiv \inf\{|\zeta|_2 : \zeta \in A^{\alpha}T\}$. Since $dT'' + f(T) \in A^{\alpha}T$ for all $T \in \mathcal{D}^{\alpha}$, we have that

$$|\overset{\circ}{A}{}^{\alpha}S^{\alpha}(t)\phi|_2 \leq \left[\int_0^{\rho} |dT''(x) + f(T(x))|^2 dx\right]^{\frac{1}{2}}$$

$$+ \frac{1}{t}\left[\int_0^{\rho} |\phi(x) - T(x)|^2 dx\right]^{\frac{1}{2}}$$

for all $t > 0$ and $T \in \mathcal{D}^{\alpha}$. Taking $T(x) \equiv 0$ if $\alpha < \infty$ and $T(x) \equiv c_o(\rho - x)/\rho$ if $\alpha = \infty$ and using the fact that f is bounded on $[0,b_o]$ justifies

$$|\overset{\circ}{A}{}^{\alpha}S^{\alpha}(t)\phi|_2 \leq \overline{M}(1 + \frac{1}{t}) \quad \text{for all} \quad t > 0 \quad \text{and} \quad \phi \in \mathcal{D} \tag{3.15}$$

where $\overline{M} > 0$ is a constant. If

$$u^{\alpha}(\cdot,t) \equiv S^{\alpha}(t)\Phi \quad \text{for} \quad t \geq 0, \quad \Phi \in \mathcal{D} \tag{3.16}$$

and if $\gamma^{\alpha}(t) \in [0,\rho]$ such that $u^{\alpha}(x,t) = 0$ for $\gamma^{\alpha}(t) \leq x \leq \rho$, then we have that

$$[\overset{\circ}{A}{}^{\alpha}S^{\alpha}(t)\Phi](x) = du^{\alpha}_{xx}(x,t) + f(u^{\alpha}(x,t)) \quad \text{for}$$

$$\text{all} \quad t > 0 \quad \text{and almost all} \quad x \in [0,\gamma^{\alpha}(t)].$$

Therefore, since $u^{\alpha}_{xx}(x,t) \equiv 0$ for $\gamma^{\alpha}(t) \leq x \leq \rho$,

$$\left[\int_0^{\rho} |du^{\alpha}_{xx}(x,t)|^2 dx\right]^{\frac{1}{2}} = \left[\int_0^{\gamma^{\alpha}(t)} |du^{\alpha}_{xx}(x,t)|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^{\gamma^{\alpha}(t)} |du^{\alpha}_{xx}(x,t) + f(u^{\alpha}(x,t))|^2 dx\right]^{\frac{1}{2}}$$

$$+ \left[\int_0^{\gamma^{\alpha}(t)} |f(u^{\alpha}(x,t))|^2 dx\right]^{\frac{1}{2}}$$

$$\leq |\overset{\circ}{A}{}^{\alpha}S^{\alpha}(t)\Phi|_2 + M_1$$

where $M_1 \leq \rho^{\frac{1}{2}}|f(b_o)|$. Combining this estimate with (3.15) shows that there is a constant $Q > 0$ (independent of $\Phi \in \mathcal{D}$ and $t > 0$) such that

$$\left.\begin{array}{l} \left[\int_0^{\rho} |u^{\alpha}_{xx}(x,t)|^2 dx\right]^{\frac{1}{2}} \leq Q(1 + \frac{1}{t}) \text{ for all } t > 0 \text{ and for all} \\ \text{solutions } u^{\alpha} \text{ to (2.1) such that } u^{\alpha}(\cdot,0) \in \mathcal{D}. \end{array}\right\} \tag{3.17}$$

Since the members of $\{f_k\}_1^{\infty}$ are uniformly bounded on $[0,b_o]$, we can show in similar fashion that Q can be chosen so that

$$\left[\int_0^\rho |\partial_{xx} w_k^\alpha(x,t)|^2 dx\right]^{\frac{1}{2}} \leq Q(1 + \frac{1}{t}) \quad \text{for all } t > 0 \quad \text{and} \tag{3.18}$$

all solutions w_k^α to (2.14) such that $w_k^\alpha(\cdot,0) \in \mathcal{D}$.

In particular, Q is independent of $k \geq 1$.

Proposition 3.6. Suppose $\Phi \in \mathcal{D}$ and u^α is defined by (3.16). Then $u_x^\alpha(x,t)$ exists for all $(x,t) \in [0,\rho] \times (0,\infty)$ and there is a constant $Q > 0$ (independent of $\Phi \in \mathcal{D}$) such that

$$|u_x^\alpha(x,t) - u_x^\alpha(y,t)| \leq Q(1 + \frac{1}{t})|x - y|^{\frac{1}{2}} \quad \text{for} \tag{3.19}$$

all $t > 0$ and $x,y \in [0,\rho]$.

Further, for each $0 < \delta < R$ there are constants $P = P(\delta,R) > 0$ and $\nu = \nu(\delta,R) \in (0,1)$ such that

$$|u^\alpha(x,t) - u^\alpha(y,s)| \leq P(|x - y|^\nu + |t - s|^\nu) \tag{3.20}$$

for all $\delta \leq t,s \leq R$ and $x,y \in [0,\rho]$.

Proof. If $\{u^\alpha,\gamma^\alpha\}$ is the solution to (2.1) with $u^\alpha(\cdot,t) \in \mathcal{D}$, then Hölder's inequality (along with the fact that $u^\alpha(\cdot,t)$ is in $\mathcal{D}^\alpha$ for $t > 0$) shows that

$$|u_x^\alpha(x,t) - u_x^\alpha(y,t)| \leq \int_y^x |u_{xx}^\alpha(x,t)| dx$$

$$\leq \left[\int_y^x dx\right]^{\frac{1}{2}} \left[\int_y^x |u_{xx}^\alpha(x,t)|^2 dx\right]^{\frac{1}{2}} \leq |x - y|^{\frac{1}{2}} Q(1 + \frac{1}{t})$$

for all $t > 0$ and $0 \leq y < x < \rho$. Thus (3.19) is valid. Using (3.18) instead of (3.17), we see that if $w_k^\alpha(\cdot,t) \equiv S_k^\alpha(t)\Phi$ for $t \geq 0$, then

$$\left.\begin{array}{l} |\partial_x w_k^{\alpha}(z,t) - \partial_x w_k^{\alpha}(v,t)| \leq |z - v|^{\frac{1}{2}} Q(1 + \frac{1}{t}) \\ \text{for all } t > 0,\ z,v \in [0,\rho] \text{ and } k = 1,2,\ldots \end{array}\right\} \quad (3.21)$$

In particular, it follows easily from (3.21) that

$$\left.\begin{array}{l} \sup\{|\partial_x w_k^{\alpha}(x,t)| : x \in [0,\rho], k = 1,2,\ldots\} = q(t) < \infty \\ \text{for all } t > 0 \text{ and all } w_k^{\alpha} \text{ such that } w_k^{\alpha}(\cdot,0) \in \mathcal{D}. \end{array}\right\} \quad (3.22)$$

From Theorem 6.2 (p. 457) of [5], we conclude that w_k^{α} is a classical solution to (2.14) and that w_k^{α} is Hölder continuous on $[0,\rho] \times [\delta,R]$ for each $0 < \delta < R$. Now suppose that $0 < \delta < R$ and consider the case $\alpha = \infty$. Since $w_k^{\infty}(0,t) \equiv c_o$, $w_k^{\infty}(\rho,t) \equiv 0$ (for $t \in [\delta,R]$), $w_k^{\infty}(x,\delta)$ is C^1 in $x \in [0,\rho]$, and $|\partial_x w_k^{\infty}(x,\delta)| \leq q(\delta)$ by (3.22), we have that $w_k^{\infty}(x,t)$ is Lipschitz continuous on

$$\{(0,t) : t \in [\delta,R]\} \cup \{(\rho,t) : t \in [\delta,R]\} \cup \{(x,\delta) : x \in [0,\rho]\}$$

with a Lipschitz constant independent of $k \geq 1$ and $\Phi \in \mathcal{D}$. Applying Theorem 1.1 (p. 419) of [5] shows that for each $0 < \delta < R$ there are constants $\rho > 0$ and $0 < \nu < 1$ (independent of k and $\Phi \in \mathcal{D}$) such that

$$\left.\begin{array}{l} |w_k^{\infty}(x,t) - w_k^{\infty}(y,s)| \leq P(|x - y|^{\nu} + |t - s|^{\nu}) \text{ for} \\ \text{all } x,y \in [0,\rho],\ \delta \leq t,s \leq R, \text{ and } k = 1,2,\ldots \end{array}\right\} \quad (3.23)$$

Letting $k \to \infty$ and using (3.13) shows that (3.20) is true for $\alpha = \infty$. The case when $\alpha = 0$ follows similarly as we note that $w_k^{o}(\cdot,t)$ extended symmetrically to $[-\rho,\rho]$ is a solution to the corresponding homogeneous Dirichlet problem on $[-\rho,\rho]$. Finally, in the case $0 < \alpha < \infty$, we use Theorem 7.4 (p. 491) of [5] to see that (3.20) holds with u^{α} replaced by w_k^{α}. Then by Theorem 7.2 (p. 486) of [5] we conclude that there exist P

and ν for (3.20) [with u^{α} replaced by w_k^{α}] which are independent of $k \geq 1$ and $\phi \in \mathcal{D}$. So, again, (3.20) follows with similar argument and the proof is complete.

We note that Proposition 3.6 verifies (2.17) and (2.18). Further, since S_k^{α} satisfies properties (P1) - (P4) of Lemma 2.5, the rest of Proposition 2.6 is immediate using Lemma 3.4. We transition now to the subject of invariant sets for the rest of this chapter. As a consequence of Theorem 3.5, we can show that the set $\hat{\mathcal{D}}$ is invariant with respect to the nonlinear semigroup $S^{\alpha} = \{S^{\alpha}(t) : t \geq 0\}$. Recall that

$$\begin{aligned} \hat{\mathcal{D}} = \{\phi \in \mathcal{D} &: \text{there exists a real number } r = r(\phi) \text{ such that} \\ &\phi \text{ is nondecreasing on } [0,r] \text{ and nonincreasing on } [r,\rho]\}. \end{aligned} \tag{3.24}$$

Let $\sigma > 0$ satisfy $\sigma < \rho$ for our next lemma.

<u>Lemma 3.7</u>. Let $\overline{\Phi} \in \hat{\mathcal{D}}$ such that $\overline{\Phi}$ is continuously differentiable on $[0,\sigma]$. Also suppose α is fixed such that $0 < \alpha < \infty$ and suppose $\phi(x)$ is the differentiable solution to the following problem (with $\phi \geq 0$):

$$\left.\begin{aligned} &d\phi'' + f(\phi) = \lambda\phi - \lambda\overline{\Phi}, \quad 0 < x < \sigma, \quad \lambda > 0 \\ &\phi'(0) = \alpha\phi(0), \quad \phi(\sigma) = 0 \end{aligned}\right\} \tag{3.25}$$

Then there exists $\overline{r} \in (0,\sigma)$ such that ϕ is nondecreasing on $[0,\overline{r}]$ and nonincreasing on $[\overline{r},\sigma]$.

<u>Proof</u>. Let $r \in [0,\sigma]$ be such that $\overline{\Phi}$ is nondecreasing on $(0,r)$ and nonincreasing on (r,σ). Differentiating the equation in (3.25) with respect to x, we have

$$d(\phi')'' + f'(\phi)\phi' = \lambda\phi' - \lambda\overline{\Phi}'.$$

Suppose (for contradiction) that $\phi'(x)$ reaches a negative minimum at $x = x_o \leq r$. Then

$$\left.\begin{aligned} d(\phi')''(x_o) &= [\lambda - f'(\phi(x_o))]\phi'(x_o) - \lambda\overline{\Phi}'(x_o) \\ &\leq [\lambda - f'(\phi(x_o))]\phi'(x_o) < 0 \end{aligned}\right\} \qquad (3.26)$$

by the hypothesis on $\overline{\Phi}$. But (3.26) contradicts the maximum principle and so we have that $\phi'(x)$ cannot attain a negative minimum for $x \leq r$. Now $\phi(x)$ may be either nonincreasing or increasing at $x = r$ but $\phi'(x)$ must attain a negative minimum in $(r,\sigma]$ (since $\phi(\sigma) = 0$). Next, if $\phi'(r) > 0$, let $\xi = \max\{x | \phi'(y) \geq 0 \text{ for } r \leq y \leq x\}$, and otherwise let $\xi = r$. Now suppose $\phi'(x) > 0$ for some $x \in (\xi,\sigma)$. Then since $\phi(\sigma) = 0$, $\phi'(x)$ attains a positive maximum at some $q \in (\xi,\sigma)$. But

$$\left.\begin{aligned} d(\phi')''(q) &= [\lambda - f'(\phi(q))]\phi'(q) - \lambda\overline{\Phi}'(q) \\ &\geq [\lambda - f'(\phi(q))]\phi'(q) > 0. \end{aligned}\right\} \qquad (3.27)$$

(3.27) contradicts a positive maximum occurring at $x = q$ and so $\phi'(x) \leq 0$ for $x \in (\xi,\sigma)$.

Since $\phi'(0) = \alpha\phi(0) > 0$, we know ϕ is nondecreasing in a neighborhood of $x = 0$, reaches a maximum and then is nonincreasing until it becomes zero at $x = \sigma$. This is precisely the conclusion of the lemma and the proof is complete.

Let $t = \overline{t} > 0$, let $n = N$ be fixed but arbitrary, and let

$$\phi_1(x) = [(I - \frac{\overline{t}}{N} A^\alpha)^{-1}\overline{\Phi}](x), \quad 0 < x < \sigma.$$

Since $\overline{\Phi} \in (I - \frac{\overline{t}}{N} A^\alpha)\phi_1$ and if ϕ_1 is positive on $(0,\sigma)$, then we have

$$\phi_1 - \frac{\bar{t}}{N} A^\alpha \phi_1 = \phi_1 - \frac{\bar{t}}{N} (d\phi_1'' + F\phi_1) = \bar{\Phi} \quad \text{or}$$

$$\frac{\bar{t}}{N} d\phi_1'' + \frac{\bar{t}}{N} F\phi_1 = \phi_1 - \bar{\Phi} \quad \text{for} \quad x \in [0,\sigma).$$

Thus

$$d\phi_1'' + F\phi_1 = (N/\bar{t})\phi_1 - (N/\bar{t})\bar{\Phi} \quad \text{wherever } \phi_1 \text{ is positive.}$$

So if $\bar{\Phi}$ satisfies the hypothesis of Lemma 3.7, then $\phi_1 \in \hat{\mathcal{D}}$. If

$$\phi_2 \equiv (I - \frac{\bar{t}}{N} A^\alpha)^{-2}\bar{\Phi} = (I - \frac{\bar{t}}{N} A^\alpha)^{-1}[(I - \frac{\bar{t}}{N} A^\alpha)^{-1}\bar{\Phi}] = (I - \frac{\bar{t}}{N} A^\alpha)^{-1}\phi_1,$$

then, since ϕ_1 satisfies the hypothesis of $\bar{\Phi}$ in Lemma 3.7, we also have $\phi_2 \in \hat{\mathcal{D}}$. By induction, $\phi_N \in \hat{\mathcal{D}}$. Since n = N was fixed but arbitrary, we have that [for all $n \geq 1$] $(I - \frac{\bar{t}}{n} A^\alpha)^{-n}\bar{\Phi}$ is in $\hat{\mathcal{D}}$ whenever $\bar{\Phi} \in \hat{\mathcal{D}}$ and $\bar{\Phi}$ is continuously differentiable. Therefore $S^\alpha(\bar{t})\bar{\Phi} = \lim_{n\to\infty} (I - \frac{\bar{t}}{n} A^\alpha)^{-n}\bar{\Phi}$ is in $\hat{\mathcal{D}}$. Since $\bar{t}$ was arbitrary, $S^\alpha(t)\bar{\Phi} \in \hat{\mathcal{D}}$ for all $t > 0$. Finally, since continuously differentiable members of $\hat{\mathcal{D}}$ are dense in $\hat{\mathcal{D}}$ we have $S^\alpha(t)\Phi \in \hat{\mathcal{D}}$ for all $t \geq 0$ and $\Phi \in \hat{\mathcal{D}}$. (See also [9]).

<u>Proposition 3.8</u>. Let $\Phi \in \hat{\mathcal{D}}$ and α satisfy $0 < \alpha < \infty$ in (2.1). Then for each $t \geq 0$, there exists $r = r(t,\Phi) \in (0,\rho)$ such that $S^\alpha(t)\Phi$ is nondecreasing on $(0,r)$ and nonincreasing on (r,ρ). Thus, $S^\alpha(t) : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ for all $t \geq 0$.

A result needed in chapter 4 follows.

<u>Corollary 3.9</u>. With $\Phi \in \hat{\mathcal{D}}$ as the initial value, let $\{u,\gamma\}$ be the solution to (2.1) where $u(\cdot,t;\, \Phi) = S^\alpha(t)\Phi$. Then for each $t > 0$, there exists a unique real number $q = q(t) \in [0,\gamma(t))$ such that $u_x(x,t) > 0$ for $x \in (0, q)$ and such that $u_x(x,t) < 0$ for $x \in (q,\gamma(t))$.

Proof. We claim that $r = r(t,\Phi)$ guaranteed in Proposition 3.8 is the appropriate candidate here. So let $q(t) = r(t,\Phi)$ and suppose (for contradiction) there exists (x_o,t_o) such that $u_x(x_o,t_o) = 0$ and $x_o \in (q(t_o),\gamma(t_o))$. Obviously, $0 < u(x_o,t_o) < u(q(t_o),t_o)$. Thus there exist w_1 and w_2 satisfying $q(t_o) < w_1 < x_o < w_2 < \gamma(t_o)$, $u_x(w_1,t_o) < 0$, and $u_x(w_2,t_o) < 0$. Since $u_x(\cdot,t)$ is a continuous function (for $t > 0$), we can choose $\varepsilon > 0$ so that $u_x(x,t) \leq 0$ for $(x,t) \in [w_1,w_2] \times [t_o - \varepsilon,t_o]$. Let $v(x,t) = u_x(x,t)$ and then $v(w_1,t) \leq 0$, $v(w_2,t) \leq 0$ for $t \in [t_o - \varepsilon,t_o]$. Also $v(x,t_o - \varepsilon) \leq 0$. Finally, as shown in Proposition 4.3 (to come), v satisfies:

$$dv_{xx} + f'(u)v = v_t, \quad t \in (t_o - \varepsilon,t_o], \quad w_1 < x < w_2. \tag{3.28}$$

Applying Theorem 2 of the appendix, we have $v \equiv 0$ on $[w_1,w_2] \times [t_o - \varepsilon,t_o]$. This is an obvious contradiction and so the hypothesis that (x_o,t_o) exists is false. Thus in $(q(t),\gamma(t))$, $u_x(x,t)$ is negative-valued. *Using a* similar proof with $w = -v = -u_x$, it can be shown that $u_x(x,\cdot)$ is positive valued on $(0,q)$. This completes the proof.

While Proposition 3.8 is more significant, we will state a similar result relative to $\mathcal{D}$.

Corollary 3.10. Let S^α be the semigroup of nonlinear operators defined in (2.16). Then $S^\alpha(t) : \mathcal{D} \to \mathcal{D}$, i.e. the family S^α is invariant with respect to the set $\mathcal{D}$.

Proof. Let $\Phi \in \mathcal{D}$ and suppose for some pair (x_o,t_o) that $[S^\alpha(t_o)\Phi](x_o) = u(x_o,t_o) > b_o$. (Recall that $\mathcal{D} \equiv \{\phi \in L^2 : 0 \leq \phi(x) \leq b_o$ a.e. for $x \in [0,\rho]\}$). Then since $\Phi(x) \leq b_o$ a.e. on $[0,\rho]$, there exists $(\overline{x},\overline{t})$ such that $u(x,t)$ achieves a maximum at $(\overline{x},\overline{t})$ where $0 < \overline{t} \leq t_o$. If $\alpha = \infty$, then

$u(0,\bar{t}) = c_o \leq b_o$. If $0 < \alpha < \infty$, then $u_x(0,\bar{t}) > 0$. If $\alpha = 0$, $u_x(0,\bar{t}) = 0$. But for $\alpha = 0$, if $\bar{x} = 0$ then Theorem 3 of the appendix requires that $u_x(0,\bar{t}) < 0$. So for $0 \leq \alpha \leq \infty$, $\bar{x} \neq 0$. Obviously, $\bar{x} \neq \gamma(\bar{t})$ since $u(\gamma(\bar{t}),\bar{t}) = 0$. Finally, suppose $0 < \bar{x} < \gamma(\bar{t})$. Then by Theorem 2 of the appendix, if $u(\bar{x},\bar{t}) = M$, then $u(x,t) \equiv M$ for all $t \leq \bar{t}$. This is an obvious contradiction and we conclude that $S^\alpha(t)\phi \in \mathcal{D}$ for all $t \geq 0$ and all $\phi \in \mathcal{D}$. This completes the proof.

Now that we have shown $\hat{\mathcal{D}}$ and $\mathcal{D}$ to be invariant with respect to the family $S^\alpha = \{S^\alpha(t) : t \geq 0\}$, we continue here with the last invariant set to be discussed. Let

$$\mathcal{D}^- \equiv \{\phi \in \mathcal{D} : \phi \text{ is nonincreasing on } [0,\rho]\}.$$

Note that $\mathcal{D}^-$ satisfies $\mathcal{D}^- \subset \hat{\mathcal{D}} \subset \mathcal{D}$. Similar to Lemma 3.7, we now consider a boundary value problem with σ satisfying $0 < \sigma < \rho$.

<u>Lemma 3.11</u>. Let $\bar{\Phi} \in \mathcal{D}^-$ such that $\bar{\Phi}$ is continuously differentiable on $[0,\sigma]$ and $\bar{\Phi}(0) \leq c_o$. Also suppose ϕ is the differentiable solution to the following problem (with $\phi \geq 0$):

$$\left.\begin{aligned} &d\phi'' + f(\phi) = \lambda\phi - \lambda\bar{\Phi}, \quad 0 < x < \sigma, \quad \lambda > 0 \\ &\phi'(0) = \alpha\phi(0), \quad \phi(\sigma) = 0 \end{aligned}\right\} \tag{3.29}$$

Then, for $\alpha = 0$ or $\alpha = \infty$, ϕ is nonincreasing on $[0,\sigma]$.

<u>Proof</u>. This proof is similar to that of Lemma 3.7. We first differentiate the equation in (3.29) with respect to x which yields

$$d(\phi')'' + f'(\phi)\phi' = \lambda\phi' - \lambda\bar{\Phi}'$$

Suppose (for contradiction) that $\phi'(x)$ has a positive maximum at $x = x_o \in [0,\sigma]$. Since ϕ is nonnegative on $[0,\sigma]$, $x_o \neq \sigma$. Next, suppose $0 \leq x_o < \sigma$ and so

$$d(\phi')''(x_o) = [\lambda - f'(\phi(x_o))]\phi'(x_o) - \lambda\bar{\Phi}'(x_o) > -\lambda\bar{\Phi}'(x_o) \geq 0. \tag{3.30}$$

However, this contradicts the assumption that a maximum occurs at $x = x_o$. Last since $\phi'(0) = 0$ when $\alpha = 0$, $x_o \neq 0$ in the Neumann case. If $\alpha = \infty$, then by (3.29) and by the hypothesis on $\bar{\Phi}$, $d\phi''(0) > 0$. But then $\phi'(x)$ cannot be maximum at $x = 0$ if $\phi''(0) > 0$ and we have a final contradiction. Thus $\phi'(x) \leq 0$ for $0 \leq x \leq \sigma$ ($\alpha = 0$ or $\alpha = \infty$) and the lemma is proved.

Using Lemma 3.11 we can prove the following:

Proposition 3.12. Let $\alpha = 0$ or $\alpha = \infty$. Also suppose $\Phi \in \mathcal{D}^-$ where $\Phi(0) \leq c_o$. Then for each $t > 0$, $S^\alpha(t)\Phi$ is nonincreasing on $[0,\rho]$. In other words, $\mathcal{D}^-$ is invariant relative to the families S^o and S^∞ when $c_o = b_o$.

Proof. Using (3.14) of Theorem 3.5, we have that $S^\alpha(t)\bar{\Phi}$ is nonincreasing, since for each integer n, the function $(I - \frac{t}{n} A^\alpha)^{-n}\bar{\Phi}$ is nonincreasing where positive (here again $\bar{\Phi}$ is assumed continuously differentiable). Since continuously differentiable members of $\mathcal{D}^-$ are dense in $\mathcal{D}^-$, we conclude $S^o(t)\Phi, S^\infty(t)\Phi \in \mathcal{D}^-$ for all $t \geq 0$ and $\Phi \in \mathcal{D}^-$.

Remark. Indeed, Proposition 3.12 may be improved since it is the case that for each $t > 0$, $[S^o(t)\Phi](x)$ and $[S^\infty(t)\Phi](x)$ have non-zero x-partial derivatives on $(0,\gamma(t))$. This is a straightforward consequence of Corollary 3.9 in which $q = 0$ for $\alpha = 0$ or $\alpha = \infty$.

4. CONTINUITY AND MONOTONE PROPERTIES OF THE MOVING BOUNDARY

There are two major objectives in this chapter. First, we will prove that the moving boundary $\gamma(t)$ is a continuous function of time for any α satisfying $0 \leq \alpha \leq \infty$ and for initial values Φ in $\hat{\mathcal{D}}$. Second, we discuss sufficient conditions which will make $\gamma(t)$ either strictly decreasing or strictly increasing. Before proving continuity, we need to state and prove some necessary lemmas.

Lemma 4.1. If $\gamma(t)$ is the moving boundary in (2.1) (existing by Theorem 2.1), then $\gamma(t)$ is lower semicontinuous.

Proof. Suppose $\{\tau_n\}_1^\infty$ is a sequence of values of t converging to T and suppose $\underline{\lim}\ \gamma(\tau_n) = \sigma$. Since a subsequence $\{\tau_{n_j}\}_1^\infty$ of $\{\tau_n\}_1^\infty$ exists such that $\lim \{\gamma(\tau_{n_j})\} = \sigma$, we consider only $\{\tau_{n_j}\}_1^\infty$. Let x_o be any value satisfying $\sigma < x_o < \rho$. Since $\lim \gamma(\tau_{n_j}) = \sigma$, there exists $N = N(x_o)$ such that for $n_j \geq N$, $\gamma(\tau_{n_j}) < x_o$. This implies for $n_j \geq N(x_o)$ that $u(x_o,\tau_{n_j}) = 0$. Since u is continuous in t by Proposition 2.6, $u(x_o,T) = \lim\limits_{n_j\to\infty} u(x_o,\tau_{n_j}) = 0$. This is true for all x_o such that $x_o > \sigma$. But then we have $\gamma(T) \leq \sigma = \underline{\lim}\ \gamma(\tau_n)$ and the lemma is proved.

In order to utilize some existing theory, we need the following continuity result.

Lemma 4.2. Let G be any closed rectangular region [in the support of u(x,t)] which intersects neither the moving boundary $\gamma(t)$ nor the line $t = 0$. Also suppose f is locally Hölder continuous (for $u > 0$)

with exponent $\frac{\lambda}{\nu}$, $\lambda \leq \nu$. Then the function $u(x,t)$ in (2.1) is in $C^{2+\lambda}(G)$ (i.e. u_t, u_{xx} are Hölder continuous on G) whenever u is jointly Hölder continuous in x and t with exponent ν.

Proof. The conclusion follows from Friedman (see Theorem 9 (p. 69) of [3]). So we need only check that the hypothesis of Friedman's theorem is satisfied and we begin with a condition on the function f. By the general hypothesis that f' is Hölder continuous, there exists $M < \infty$ such that

$$|f(\xi_1) - f(\xi_2)| \leq M|\xi_1 - \xi_2|^{\lambda/\nu} \text{ for } 0 \leq \xi_1,\xi_2 \leq b_o.$$

From (2.18) recall that for $0 < \delta < R < \infty$

$$|u(x_1,t_1) - u(x_2,t_2)| \leq P(|x_1 - x_2|^{\nu} + |t_1 - t_2|^{\nu}) \quad \text{for}$$
$$0 \leq x_1,x_2 \leq \rho \quad \text{and for} \quad 0 < \delta \leq t_1,t_2 \leq R$$

and if

$$\hat{f}(x,t) \equiv f(u(x,t))$$

then we have

$$\begin{aligned}|\hat{f}(x_1,t_1) - \hat{f}(x_2,t_2)| &\leq MP^{\lambda/\nu}(|x_1 - x_2|^{\nu} + |t_1 - t_2|^{\nu})^{\lambda/\nu} \\ &\leq MP^{\lambda/\nu}(|x_1 - x_2|^{\lambda} + |t_1 - t_2|^{\lambda}) \quad \text{for} \\ &0 \leq x_1,x_2 \leq \rho \text{ and for } 0 < \delta \leq t_1,t_2 \leq R.\end{aligned}$$

This implies $\hat{f}$ is locally Hölder continuous (exponent λ) in G. Further, since $u > 0$ on the boundary of G, $\hat{f}(x,t)$ is bounded on G and we can disregard any possible growth condition on $\hat{f}(x,t)$ (as Friedman allows in his theorem).

Next since G is rectangular, a side s of G corresponding to a constant value of x obviously has the outside strong sphere property (see p. 69 of [3]). Thus s has local barriers with respect to the operator B, where $Bu \equiv u_{xx} - u_t$. Finally, we know that u is continuous on $[0,\rho] \times [0,\infty)$ and so u is continuous on the boundary of G. Therefore, the hypothesis of Friedman's theorem is satisfied and we conclude that u(x,t) is in $C^{2+\lambda}$ on G. This completes the proof of the lemma.

Note that in Lemma 4.2 we do not discuss uniqueness. From Theorem 2.1, we know the solution to (2.1) is unique. Now let z(x,t) = u(x,t) for $(x,t) \in \partial G$ (the boundary of G). Then the solution of

$$\left.\begin{aligned} dw_{xx} - w_t &= \hat{f}(x,t), \quad (x,t) \in \dot{G} \\ w(x,t) &= z(x,t), \quad (x,t) \in \partial G \end{aligned}\right\} \tag{4.1}$$

must agree with the solution of (2.1) restricted to G.

Now that we have shown u(x,t) to be in $C^{2+\lambda}$, we reference Friedman (see p. 72 of [3]) to conclude that u(x,t) is somewhat smoother than the function f itself. Specifically if f'(u) is Hölder continuous (exponent ν) in a region $\dot{G}$ (as described in Lemma 4.2) and since, by Proposition 2.6, $u_x(x,t)$ is Hölder continuous (exponent ν) on $[0,\rho] \times [\delta,R]$, then the product $\frac{df}{du} \cdot u_x$ is also Hölder continuous. If so, then u_{xxx} and u_{xt} exist and are Hölder continuous.

With the above paragraph in mind, we may differentiate the equation in (2.1) with respect to x. Such existence is necessary so that we may prove the following result.

Proposition 4.3. Suppose that $f'(u)$ is Hölder continuous (i.e. f is in $C^{1+\nu}$ for $0 < \nu < 1$) on the interval $[0,b_o]$. Suppose $\phi \in \hat{\mathcal{D}}$ is nonconstant and differentiable, let $\delta > 0$, and consider the following problems:

$$\left.\begin{array}{lll} d(u_x)_{xx} + f'(u)u_x = (u_x)_t, & x_1 < x < x_2, & t > 0 \\ u_x(x_1,t) = h_1(t) \le 0, & u_x(x_2,t) = h_2(t) \le 0, & t \ge 0 \\ u_x(x,0) = \Phi'(x) \le 0, & x_1 < x < x_2 & \end{array}\right\} \tag{4.2}$$

and

$$\left.\begin{array}{lll} dw_{xx} + f'(u)w = w_t, & x_1 < x < x_2 & t > 0 \\ w(x_1,t) = 0,\ w(x_2,t) = 0, & t \ge 0 & \\ w(x,0) = \Phi'(x) \le 0, & x_1 < x < x_2 & \end{array}\right\} \tag{4.3}$$

Then $u_x \le w \le 0$.

Proof. Let $v = e^{-\delta t}u_x$ and $z = e^{-\delta t}w$. We show that $v \le z \le 0$ and then multiplication by $e^{\delta t}$ yields the conclusion of the proposition. First, we substitute into the differential equations of (4.2) and (4.3):

$$de^{\delta t}v_{xx} + f'(u)e^{\delta t}v = \delta e^{\delta t}v + e^{\delta t}v_t \tag{4.4}$$

and

$$de^{\delta t}z_{xx} + f'(u)e^{\delta t}z = \delta e^{\delta t}z + e^{\delta t}z_t. \tag{4.5}$$

Next consider the difference $s(x,t) = v(x,t) - z(x,t)$. Now from the hypothesis we have

$$s(x,0) = 0, \quad s(x_1,t) = v(x_1,t) - z(x_1,t) = e^{-\delta t}h_1(t) \le 0, \quad \text{and}$$

$$s(x_2,t) = v(x_2,t) - z(x_2,t) = e^{-\delta t}h_2(t) \le 0.$$

If $s > 0$ then $v > z$ somewhere in $(x_1,x_2) \times (0,T]$ for $T > 0$. Assume (for contradiction) that

$$\sup_{\substack{x_1<x<x_2 \\ 0<t\le T}} \{s(x,t)\} = s(x_o,t_o) > 0.$$

Subtracting (4.5) from (4.4) gives

$$d(v - z)_{xx} + (f'(u) - \delta)(v - z) = (v - z)_t \quad \text{or}$$

$$ds_{xx} + (f'(u) - \delta)s = s_t.$$

In particular since we assume a maximum occurs at (x_o,t_o) we have

$$ds_{xx}(x_o,t_o) \ge [\delta - f'(u(x_o,t_o))]s(x_o,t_o).$$

But then $s_{xx}(x_o,t_o) > 0$ since $f'(u) \le 0$ and we reach a contradiction to the (weak) maximum principle. So it must be the case that $s \le 0$ which implies $v \le z$. Finally, $w \le 0$ because otherwise w must attain a positive maximum in $(x_1,x_2) \times (0,T]$ and, again, the (weak) maximum principle is contradicted. Thus the conclusion of the proposition is proved.

Lemma 4.4. Suppose that $T > 0$, that $0 < \gamma(T) < \rho$, and that $x_o \in (\gamma(T),\rho)$. Further suppose that, for $\delta > 0$, $u_x(\gamma(T) - \delta,T) < 0$ and suppose that for $\varepsilon_o > 0$, $u_x(\gamma(T) - \delta,t) < 0$ when $t \in [T - \varepsilon_o,T]$. Let $0 < \varepsilon < \min\{\varepsilon_o,T,\rho - \gamma(T)\}$. Then there exists $(y,\tau) \in [\gamma(T),x_o] \times [T - \varepsilon,T)$ such that $u(y,\tau) = 0$.

Proof. Set $R = [\gamma(T),x_o] \times [T - \varepsilon,T)$. Suppose (for contradiction) that $u(x,t) > 0$ for all $(x,t) \in R$. Let $z(x,t) = u_x(x,t)$ and then z satisfies

$$dz_{xx} + f'(u)z = z_t, \quad \gamma(T) < x < x_o, \quad T - \varepsilon < t < T$$

$$z(\gamma(T),t) < 0, \; z(x_o,t) \leq 0, \quad T - \varepsilon < t < T \tag{4.6}$$

$$z(x,T - \varepsilon) < 0, \quad \gamma(T) < x < x_o.$$

Now in the hypothesis, δ exists by Corollary 3.9. Further, ε_o exists because, for $u > 0$, $u_x(\cdot,t)$ is continuous. Of course, $z(\gamma(T),t) < 0$ for $t \in (T - \varepsilon,T)$ again due to Corollary 3.9.

So if w is the solution to

$$dw_{xx} + f'(u)w = w_t, \quad (x,t) \in R$$

$$w(\gamma(T),t) = 0, \quad w(x_o,t) = 0 \;\text{ for } T - \varepsilon < t < T \tag{4.7}$$

$$w(x,T - \varepsilon) = u_x(x, T - \varepsilon) < 0, \quad \gamma(T) < x < x_o$$

then by Proposition (4.3), $z \leq w \leq 0$. However, by hypothesis $z(x,T) = 0$ for $\gamma(T) \leq x \leq x_o$, and this implies $w(x,T) = 0$ for $\gamma(T) \leq x \leq x_o$. Then by backward uniqueness (see Theorem 4 of the appendix), $w \equiv 0$ on R. In particular, this contradicts $w(x,T - \varepsilon) < 0$ for $\gamma(T) < x < x_o$. Thus we conclude that the pair (y,τ) exists in R such that $u(y,\tau) = 0$ and the proof is complete.

As a final preliminary lemma we have the following:

Lemma 4.5. Suppose that $T > 0$, that $0 < \gamma(T) < \rho$, and that $\eta > 0$ is such that $\sqrt{-2d\eta/f(0)} < \rho - \gamma(T)$. Since $u(\gamma(T),T) = 0$ and u is continuous, select $T_1 > T$ such that for $T < t \leq T_1$, both $u(\gamma(T),t) \leq \eta$ and $u_x(\gamma(T),t) \leq 0$. Then

$$\gamma(t) \leq \gamma(T) + \sqrt{-2d\eta/f(0)} \quad \text{for} \quad t \in [T,T_1]. \tag{4.8}$$

Proof. For comparison purposes, consider the problem

$$\left.\begin{aligned} &dw'' - m = 0, \quad \gamma(T) \le x \le \sigma < \rho \\ &w(\gamma(T)) = \eta \\ &w(\sigma) = w'(\sigma) = 0 \end{aligned}\right\} \tag{4.9}$$

where $m > 0$ is chosen large enough so that $0 < m < -f(0)$ and $\sigma \equiv \gamma(T) + \sqrt{2d\eta/m} < \rho$. The solution to (4.9) is trivially

$$w(x) = \begin{cases} m(\sigma - x)^2/2d & \gamma(T) \le x \le \sigma \\ 0 & \sigma \le x \le \rho \end{cases} \tag{4.10}$$

We will use the maximum principle to show that $u(x,t) \le w(x)$ for all $T \le t \le T_1$ and $\gamma(T) \le x \le \rho$. Suppose (for contradiction) that there exists $(\bar{x},\bar{t}) \in [\gamma(T),\rho] \times [T,T_1]$ such that

$$u(\bar{x},\bar{t}) - w(\bar{x}) = \max\{u(x,t) - w(x) : T \le t \le T_1, \gamma(T) \le x \le \rho\} > 0$$

Clearly $\bar{x} > \gamma(T)$ by the choice of T_1 and since $u(x,T) \equiv 0$ for $x > \gamma(T)$, we have $\bar{t} > T$. Also since $u(\bar{x},\bar{t}) > 0$, we know that $\gamma(T) < \bar{x} < \gamma(\bar{t})$ which implies both $u_t(\bar{x},\bar{t})$ and $u_{xx}(\bar{x},\bar{t})$ exist. With the presumption that a maximum occurs at $(\bar{x},\bar{t})$, we see that $[u - w]_t(\bar{x},\bar{t}) = u_t(\bar{x},\bar{t}) \ge 0$. Now there are two possibilities for $\bar{x}$. First, if $\gamma(T) < \bar{x} < \sigma$, then

$$\begin{aligned} 0 \le u_t(\bar{x},\bar{t}) &= du_{xx}(\bar{x},\bar{t}) + f(u(\bar{x},\bar{t})) - dw''(\bar{x}) + m \\ &= d[u - w]_{xx}(\bar{x},\bar{t}) + f(u(\bar{x},\bar{t})) + m. \end{aligned}$$

But then by the choice of m,

$$d[u - w]_{xx}(\bar{x},\bar{t}) \ge -m - f(u(\bar{x},\bar{t})) \ge -m - f(0) > 0$$

which is a contradiction. Second, suppose that $\sigma \le \bar{x} < \gamma(\bar{t})$. Then $w'(\bar{x}) = 0$ and so

$$u_x(\bar{x},\bar{t}) = [u - w]_x(\bar{x},\bar{t}) = 0$$

since $(\bar{x},\bar{t})$ is a local maximum for $(u - w)$. However, with T_1 as chosen in the hypothesis and since $\sigma \le \bar{x} < \gamma(\bar{t})$, $u_x(\bar{x},\bar{t}) < 0$ and we again have a contradiction. Thus $u \le w$ and, since m may be taken arbitrarily close to $-f(0)$, we conclude that (4.8) is true and the lemma is proved.

At this point we state our main continuity result.

Theorem 4.6. Suppose for $t \in (T_1,T_2)$ that $\gamma(t) \le \xi < \rho$. Then the moving boundary $\gamma(t)$ is continuous on (T_1,T_2).

Proof. Suppose (for contradiction) that γ is not continuous at $T \in (T_1,T_2)$. Then there exists some $\varepsilon > 0$ and a sequence $\{t_n\} \in (T_1,T_2)$ such that (for all $n \ge 1$) $|\gamma(t_n) - \gamma(T)| \ge \varepsilon$ and $t_n \to T$ as $n \to \infty$. Due to Lemma 4.1, we may assume without loss of generality that

$$\gamma(t_n) \ge \gamma(T) + \varepsilon \quad \text{for all } n \ge 1. \tag{4.11}$$

Next choose $\eta > 0$ such that $\sqrt{-2d\eta/f(0)} < \varepsilon$ and such that $\gamma(T) + \eta < \rho$. Since $u(x,t)$ is continuous at $(\gamma(T),T)$, let $\delta > 0$ be such that $\delta < \eta$ and

$$|u(x,t)| \le \eta \quad \text{if} \quad |t - T|, \ |x - \gamma(T)| \le \delta.$$

By Lemma 4.4, there exists (y,t_o) such that

$$T - \delta \le t_o < T, \quad \gamma(T) \le y \le \gamma(T) + \delta, \quad \text{and} \quad u(y,t_o) = 0.$$

Then $\gamma(t_o) \le y < \rho$ because $\gamma(T) + \delta < \gamma(T) + \eta < \rho$. Applying Lemma 4.5 with $T_1 = T + \delta$ we have that

$$\gamma(t) \le \gamma(T) + \sqrt{-2d\eta/f(0)} < \gamma(T) + \varepsilon \tag{4.12}$$

for all $t \in [t_o, T + \delta]$. (Note that if $t_o \le t \le t_o + \delta$, then $|t - t_o| < \delta$ and $|y - \gamma(T)| \le \delta$ and hence $|u(y,t)| \le \eta$.) But $t_o < T$ and so $t_n \in [t_o, T + \delta]$ for n sufficiently large. Thus (4.12) contradicts (4.11) and we conclude that the sequence $\{t_n\}$ cannot exist. Therefore γ is continuous on (T_1, T_2).

Besides the continuity of the moving boundary, the other pursuit in this chapter is to identify conditions under which the moving boundary is a monotone function of time. We begin with a lemma.

Lemma 4.7. Suppose the initial value in (2.1) is $\phi_o(x) \equiv b_o$ for $0 \le x \le \rho$ where ρ satisfies $\rho > (-2dc_o/f(0))^{\frac{1}{2}}$. Then $u(\cdot,t_1;\ \phi_o) \ge u(\cdot,t_2;\ \phi_o)$ for $0 \le t_1 \le t_2 < \infty$.

Proof. Recall from Proposition 3.8 that $u(x,t;\ \phi_o) \in \hat{\mathcal{D}}$ for all $t \ge 0$. Then for all $h > 0$, $[S^\alpha(h)\phi_o](x) = u(x,h;\ \phi_o) \le \phi_o(x) = b_o$. By (Q2) and (Q4) of Theorem 2.1, we have (if $h = t_2 - t_1 > 0$)

$$\begin{aligned} S^\alpha(t_2)\phi_o &= S^\alpha(t_1 + t_2 - t_1)\phi_o = S^\alpha(t_1)S^\alpha(t_2 - t_1)\phi_o \\ &= S^\alpha(t_1)S^\alpha(h)\phi_o \le S^\alpha(t_1)\phi_o. \end{aligned} \tag{4.13}$$

(4.13) is precisely the conclusion of the lemma.

Remark. We observe that the requirement $\rho > (-2dc_o/f(0))^{\frac{1}{2}}$ is only necessary for the case $\alpha = \infty$ so that the interval $[0,\rho]$ properly contains the support of the steady state solution (existence of such a steady state is proved in chapter 5). Also since u_t exists everywhere in

$(0,\gamma(t)) \times (0,\infty)$, we see that $u_t(x,t;\ \Phi_o) \leq 0$ is an equivalent conclusion to Lemma 4.7.

Theorem 4.8. Suppose $u(x,t)$ of the solution pair $\{u(x,t),\gamma(t)\}$ to (2.1) satisfies $u_t \leq 0$ on $(0,\gamma(t)) \times (0,\infty)$. Then the moving boundary $\gamma(t)$ is a strictly decreasing function of time.

Proof. As with many of the other proofs, we will use the (strong) maximum principle to prove this theorem. First, from the hypothesis that $u_t \leq 0$, we observe that $\gamma(t)$ is at least nonincreasing. So we will seek a contradiction by assuming that there exists an interval $[t_1,t_2]$ and a positive number β in $(0,\rho)$ such that $u(\beta,s) = 0$ for all s in $[t_1,t_2]$. Implied here in our assumption is that for all $0 < \varepsilon \leq \frac{\beta}{2}$, $u(\beta - \varepsilon,s) > 0$ for $t_1 \leq s \leq t_2$. Now let

$$\bar{u}(x,\tau) \equiv u\left(x, \frac{t_1 + t_2}{2} + \tau\right) \quad \text{for } \tau \geq 0.$$

Also let

$$\hat{u}(x,\tau) \equiv u(x,t_1 + \tau) \quad \text{for } \tau \geq 0.$$

By hypothesis $\bar{u} \leq \hat{u}$ so that

$$v(x,t) \equiv \bar{u}(x,t) - \hat{u}(x,t) \leq 0$$

and v satisfies the following parabolic equation:

$$dv_{xx} = d\bar{u}_{xx} - d\hat{u}_{xx} = \bar{u}_t - \hat{u}_t - f(\bar{u}) + f(\hat{u}) = v_t - f(\bar{u}) + f(\hat{u}) \quad (4.14)$$

Define

$$g(x,t) = \begin{cases} \dfrac{f(\bar{u}) - f(\hat{u})}{\bar{u} - \hat{u}} & \text{if } \bar{u}(x,t) \neq \hat{u}(x,t) \\ f'(\bar{u}) & \text{if } \bar{u}(x,t) = \hat{u}(x,t) \end{cases}$$

We note that $g(x,t) \leq 0$ because of the nonincreasing nature of f. Then (4.14) may be rewritten as

$$dv_{xx} - v_t + g(x,t)v = 0.$$

By hypothesis, $v(x,t)$ satisfies the following boundary conditions for $0 \leq t \leq \frac{t_2 - t_1}{2}$:

$$v(\beta,t) = \bar{u}(\beta,t) - \hat{u}(\beta,t) = 0 \quad \text{and} \quad v_x(\beta,t) = 0.$$

Next we suppose there exists (x_o,t_o) such that $v(x_o,t_o) = 0$ for $0 \leq t_o \leq \frac{t_2 - t_1}{2}$ and $0 < x_o < \beta$. Then, by Theorem 2 of the appendix, $v \equiv 0$ for all $t \leq t_o$. In particular this implies that $u(\cdot,t_1 + t_o) = u\left(\cdot, \frac{t_1 + t_2}{2} + t_o\right)$ and, since $u_t \leq 0$, we have $u(\cdot,s) = u(\cdot,t_1 + t_o)$ and $u_t(\cdot,s) = 0$ for $t_1 + t_o \leq s \leq \frac{t_1 + t_2}{2} + t_o$. By hypothesis, $u(\beta - \varepsilon,t_1 + t_o) > 0$. To have a contradiction in the case $\alpha = \infty$, we must have that $[0,\beta]$ is not the support of the unique steady state solution for (2.1) (see chapter 5). This is reasonable since otherwise the "moving" boundary would be fixed for all time, i.e. $\gamma(t) \equiv \beta$. In the case $0 \leq \alpha < \infty$, we show in chapter 5 that the trivial solution is the unique steady state. Thus, for $0 \leq \alpha < \infty$ we also have a contradiction to the assumption that $0 < x_o < \beta$. Therefore, for $0 < x < \beta$ and $0 < t \leq \frac{t_2 - t_1}{2}$, we have $v < 0$. Finally, since $v(\beta,t) = 0$ is a maximum value for v and since $x = \beta$ is a boundary, we apply Theorem 3 of the appendix to conclude that the outward normal derivative, $v_x(\beta,t)$ is

positive. This contradicts the hypothesis that $v_x(\beta,t) = 0$. Then the original assumption that $t_2 > t_1$ is not true and so $\gamma(t)$ cannot be constant on any interval of time. In summary, given an arbitrary value of $t \geq 0$ and given $\varepsilon > 0$, there exists $\delta = \delta(t,\varepsilon) > 0$ such that $u(\gamma(t) - \delta, t + \varepsilon) = 0$. This statement says precisely that the moving boundary $\gamma(t)$ is strictly decreasing and the theorem is proved.

As a clarification of Theorem 4.8, we point out that there may be some delay before the moving boundary appears. For example, if $\lim_{x\to\rho^-} u_x(x,t) = u_x(\rho^-,t)$, then $u_x(\rho^-,t)$ may be negative for $t \in (0,\tau_o)$ and so the moving boundary begins at $t = \tau_o$. In the event of such a delay, there is a fixed boundary at $x = \rho$ for $t \in (0,\tau_o)$ and then Theorem 4.8 is valid for the moving boundary starting at $t = \tau_o$. We summarize in the next corollary.

Corollary 4.9. Suppose $\Phi \in \mathcal{D}^-$, Φ is continuously differentiable, and $\lim_{x\to\rho^-} \phi'(x) = \phi'(\rho^-) < 0$. Then when $\alpha = 0$ in (2.1) there exists a $\tau_o > 0$ such that the moving boundary $\gamma(t)$ in (2.1) starts at $t = \tau_o$ and is strictly decreasing for $t \geq \tau_o$.

Proof. In Lemma 4.10 which follows, we prove that there is some delay before a moving boundary may appear. Combining Lemma 4.10 with the fact that $u(x,T) \equiv 0$ for some $T > 0$ (see chapter 5) is confirmation that a moving boundary exists and starts at some positive time.

Consider the following parabolic problem:

$$\left.\begin{array}{lll} dw_{xx} + f(w) = w_t, & 0 < x < \rho, & t > 0 \\ w_x(0,t) = 0, & w(\rho,t) = 0, & t > 0 \\ w(x,0) = \Phi(x), & 0 \le x \le \rho & \end{array}\right\} \tag{4.15}$$

We now discuss the solution $w(x,t)$ of (4.15) in the next lemma.

Lemma 4.10. Suppose Φ satisfies the hypothesis of Corollary 4.9. Then there exists $\tau_1 > 0$ such that the solution $w(x,t)$ to (4.15) *remains non-negative* for $0 \le t \le \tau_1$.

Proof. Because f is continuously differentiable, $w_x(x,t)$ is continuous in t. Thus if $\Phi'(\rho^-) < 0$, there exists $\tau_1 > 0$ such that $w_x(\rho^-,t) < 0$ when t satisfies $0 \le t < \tau_1$ (see Theorem 13 (p. 79) of [3]). As shown in Proposition 1.3, $w(x,t)$ is a lower bound for $u(x,t)$ of (2.1) and thus the lemma is proved.

An example to accompany Lemma 4.10 is this problem:

$$\left.\begin{array}{lll} w_{xx} - 1 = u_t, & 0 \le x \le \pi, & t \ge 0 \\ w_x(0,t) = 0, & u(\pi,t) = 0 & \\ u(x,0) = \cos x/2, & 0 \le x \le \pi & \end{array}\right\} \tag{4.16}$$

The series solution is

$$u(x,t) = e^{-t/4}\cos\frac{x}{2} - \frac{16}{\pi}(1 - e^{-t/4})\cos\frac{x}{2} + \frac{16}{27\pi}(1 - e^{-9t/4})\cos\frac{3x}{2}$$

$$- \frac{16}{125\pi}(1 - e^{-25t/4})\cos\frac{5x}{2} + \ldots = e^{-t/4}\cos\frac{x}{2}$$

$$+ \sum_{n=1}^{\infty} \frac{16(-1)^n}{(2n-1)^3\pi}(1 - e^{-(2n-1)^2 t/4})\cos\frac{(2n-1)x}{2} .$$

Then

$$u_x(x,t) = -\frac{1}{2}e^{-t/4}\sin\frac{x}{2}$$

$$+\sum_{n=1}^{\infty}\frac{8(-1)^{n+1}}{(2n-1)^2\pi}(1-e^{-(2n-1)^2t/4})\sin\frac{(2n-1)x}{2}.$$

Thus

$$u_x(\pi,t) = -\frac{1}{2}e^{-t/4}+\sum_{n=1}^{\infty}\frac{8(-1)^{n+1}}{(2n-1)^2\pi}(1-e^{-(2n-1)^2t/4})(-1)^{n-1}$$

$$= -\frac{1}{2}e^{-t/4}+\sum_{n=1}^{\infty}\frac{8}{(2n-1)^2\pi}(1-e^{-(2n-1)^2t/4}).$$

Obviously, for t small enough, $u_x(\pi,t)$ will be arbitrarily close to the value $-1/2$. For example, $u_x(\pi,.04) < -.224$ is a rough estimate after truncating at the fifth series term. This bound shows that $w_x(\pi,t)$ remains negative until some value of time greater than .04. In comparison to (4.16), we consider (2.1) with $\alpha = 0$ and with f identically constant at the value -1. By Proposition 1.3, the solution to (4.16) remains below the analogous solution to (2.1). Since $w_x(\pi,t) < 0$ for $0 \le t \le .04$, $w(x,t) > 0$ for $0 \le t \le .04$ and $x < \pi$. But then $u_x(\pi,t) < 0$ for $0 \le t \le .04$ and it must be the case that the moving boundary in (2.1) with $f \equiv -1$ can begin only at some time greater than .04.

To conclude this chapter, we look at specific results for the cases $\alpha = 0$ and $\alpha = \infty$. First, since there is a unique nontrivial steady state solution to (2.1) with $\alpha = \infty$ (see chapter 5), we want to discuss solutions which lie underneath this steady state. If the initial value is $\phi_L(x) \equiv 0$ for $x \in [0,\eta]$ where $[0,\eta]$ is the support of the steady state solution, then $u(x,t;\phi_L)$ is nondecreasing in time. Now $\phi_L \in \mathcal{D}^-$ and by Proposition 3.12, $S^\infty(t) : \mathcal{D}^- \to \mathcal{D}^-$ for all $t \ge 0$. Thus $S^\infty(h)\phi_L \ge \phi_L$

for $h > 0$ and we have $S^{\infty}(t_2)\Phi_L \geq S^{\infty}(t_1)\Phi_L$ whenever $t_2 \geq t_1$. In other words, $u_t(x,t;\ \Phi_L) \geq 0$ for $(x,t) \in (0,\gamma(t)) \times (0,\infty)$. We can now state a result similar to Theorem 4.8.

Corollary 4.11. Suppose for $\alpha = \infty$ that $u_t(x,t) \geq 0$, where $u(x,t)$ is in the solution pair $\{u,\gamma\}$ for (2.1). Then the moving boundary $\gamma(t)$ is strictly increasing.

Proof. The proof construction is similar to that of Theorem 4.8. Suppose $\bar{u}$ and $\hat{u}$ are defined as in the proof of Theorem 4.8 and let $v = \hat{u} - \bar{u}$. Then $v(x,t) < 0$ for $0 < x < \beta$ and, again, a contradiction occurs because v_x should satisfy $v_x(\beta,t) > 0$ by Theorem 3 of the appendix.

While only in the case $\alpha = \infty$ can there be a strictly increasing moving boundary, we do know for both $\alpha = 0$ and $\alpha = \infty$ that there are additional criteria which will cause $u(x,t)$ to be nonincreasing in time. In this direction we begin with a lemma.

Lemma 4.12. Suppose $\Phi \in \mathcal{D}^-$ is continuously differentiable, $\Phi'(0) \leq 0$, and $d\Phi''(x) + f(\Phi(x)) \leq 0$ for $0 \leq x \leq \sigma < \rho$. Let $T(x)$ satisfy:

$$\left.\begin{aligned} &dT''(x) + f(T(x)) = \lambda T(x) - \lambda\Phi(x), \quad 0 < x < \sigma, \quad \lambda > 0 \\ &T'(0) = 0, \quad T(\sigma) = 0 \end{aligned}\right\} \tag{4.17}$$

Then $T \leq \Phi$ for $x \in [0,\sigma]$.

Proof. From the hypothesis, we have $-d\Phi'' - f(\Phi) \geq 0$ and so $d(T - \Phi)'' + f(T) - f(\Phi) \geq \lambda(T - \Phi)$. Suppose (for contradiction) that $\max_{0 \leq x \leq \sigma} \{(T - \Phi)(x)\} > 0$. Obviously, this maximum does not occur at $x = \sigma$.

Also, if a maximum occurs at $x = 0$ then $(T - \Phi)'(0) \le 0$ yet by hypothesis $T'(0) - \Phi'(0) = -\Phi'(0) \ge 0$. So $(T - \Phi)'(0) = 0$ but

$$d(T - \Phi)''(0) \ge \lambda[T(0) - \Phi(0)] + f(\Phi(0)) - f(T(0))$$

$$\ge \lambda T(0) - \lambda\Phi(0) > 0.$$

This contradicts the maximum of $T - \Phi$ occurring at $x = 0$. Finally, if the maximum occurs in $(0,\sigma)$, then (at the maximum) $d(T - \Phi)'' = \lambda(T - \Phi) + f(\Phi) - f(T) \ge \lambda(T - \Phi) > 0$ which is a contradiction. Thus $\max_{0\le x\le\sigma} \{(T(x) - \Phi(x)\} \le 0$ and the proof is complete.

Now we are in position to state a condition about the initial value Φ in (2.1) which causes the solution $u(x,t;\Phi)$ to be nonincreasing in time. $\Phi \in \mathcal{D}$ satisfies this condition if

$$\Phi'(0) \le 0 \quad \text{and} \quad d\Phi''(x) + f(\Phi(x)) \le 0. \tag{4.18}$$

We use condition (4.18) in the hypothesis of the next proposition.

Proposition 4.13. If $\Phi \in \mathcal{D}^-$ satisfies (4.18) and if $\alpha = 0$ in (2.1) then $u(x,t;\Phi)$ satisfies $u_t \le 0$ for $(x,t) \in (0,\gamma(t)) \times (0,\infty)$.

Proof. Recall from Theorem 3.5 that

$$S^o(t)\Phi = \lim_{n\to\infty} (I - \frac{t}{n} A^o)^{-n}\Phi \tag{4.19}$$

for each $t > 0$. Since $A^o\phi = \{d\phi'' + F\phi\}$ for $\phi \in \mathcal{D}^o$ where

$$\mathcal{D}^o = \{\phi \in \mathcal{D} : \phi \text{ and } \phi' \text{ are absolutely continuous},$$

$$\phi'' \in L^2, \phi'(0) = 0 \text{ and } \phi(\rho) = 0\},$$

we see that the solution of (4.17) is in $\mathcal{D}^o$. From Lemma 4.12 with $\lambda = \frac{t}{n}$, we have $(I - \frac{t}{n} A^o)^{-1}\Phi \leq \Phi$. Further, we know by (3.10) that the operator $(I - \frac{t}{n} A^o)^{-1}$ preserves order. Thus

$$(I - \frac{t}{n} A^o)^{-n}\Phi \leq (I - \frac{t}{n} A^o)^{-n+1}\Phi \leq \ldots \leq (I - \frac{t}{n} A^o)^{-2}\Phi$$
$$\leq (I - \frac{t}{n} A^o)^{-1}\Phi \leq \Phi$$

and we conclude that order is preserved in the limit

$$S^o(h)\Phi = \lim_{n\to\infty} (I - \frac{h}{n} A^o)^{-n}\Phi \leq \Phi \quad \text{for all} \quad h > 0. \tag{4.20}$$

Using (Q2) and (Q4) of Theorem 2.1 and for $h > 0$, we have

$$\begin{aligned}[S^o(t + h)\Phi](x) - [S^o(t)\Phi](x) &= [S^o(t)S^o(h)\Phi](x) - [S^o(t)\Phi](x) \\ &\leq [S^o(t)\Phi](x) - [S^o(t)\Phi](x) = 0 \quad \text{for} \\ &(x,t) \in [0,\gamma(t)) \times (0,\infty).\end{aligned} \tag{4.21}$$

Multiplying (4.21) by h^{-1} and then taking the limit as $h \to 0^+$ implies that $u_t \leq 0$ for all $t > 0$, $0 < x < \gamma(t)$ and the proposition is proved.

As a result of Proposition 4.13 and Theorem 4.8, we know for $\alpha = 0$, $\Phi \in \mathcal{D}^-$ and Φ satisfying (4.18) that the moving boundary $\gamma(t)$ (in the solution pair $\{u(x,t),\gamma(t)\}$) is strictly decreasing. To prove the same result for $\alpha = \infty$, we need only require additionally that $\Phi(0) \geq c_o$ in the hypothesis of Lemma 4.12. Then if in (4.17) the boundary condition is changed to $\phi(0) = c_o$, the conclusions of Lemma 4.12 and Proposition 4.13 are valid for the case $\alpha = \infty$.

5. ASYMPTOTIC BEHAVIOR

In this chapter we concentrate on the critical points (or steady state solutions) of problem (2.1). The most interesting contrast is that for $0 \leq \alpha < \infty$, the zero function is the critical point and it is reached in finite time; and for $\alpha = \infty$, there is a unique nontrivial critical point which can not be reached in finite time. We begin with a symmetry result.

Lemma 5.1. Suppose that $\Phi \in \mathcal{D}^-$ and that

$$\Phi_s(x) \equiv \begin{cases} \Phi(x) & 0 \leq x \leq \rho \\ \Phi(-x) & -\rho \leq x \leq 0 \end{cases} \tag{5.1}$$

Next consider the following parabolic problem with f_k satisfying (2.6):

$$\left.\begin{array}{lll} dv_{xx}+f_k(v) = v_t, & -\rho < x < \rho, & t > 0 \\ v(-\rho,t) = v(\rho,t) = 0, & t > 0 & \\ v(x,0) = \Phi_s(x), & -\rho \leq x \leq \rho & \end{array}\right\} \tag{5.2}$$

Then $v(x,t) = v(-x,t)$ for $x \in [0,\rho]$ and $t > 0$.

Proof. Let $y = -x$ and define $w(y,t) \equiv v(x,t)$ for $x \in [-\rho,0]$ and $t > 0$. Then $v_x = w_y \cdot \frac{dy}{dx} = -w_y$ and $v_{xx} = -w_{yy} \cdot \frac{dy}{dx} = w_{yy}$. So for $y \in [0,\rho]$ we have

$$\begin{array}{l} dw_{yy}(y,t)+f_k(w(y,t)) = w_t(y,t), \quad 0 < y < \rho, \quad t > 0 \\ w(0,t) = v(0,t), \quad w(\rho,t) = v(\rho,t) = 0 \\ w(y,0) = \Phi(x), \quad 0 < y < \rho \end{array} \tag{5.3}$$

Now we assume (for contradiction) that for some $t = \bar{t}$, $w(\cdot,\bar{t}) \neq v(\cdot,\bar{t})$ on $[0,\rho]$. First, suppose $\inf\limits_{\substack{y\in[0,\rho] \\ t>0}} \{[w(y,t) - v(y,t)]e^{-\delta t}\} < 0$ where $\delta > 0$.

After subtracting differential equations in (5.2) and (5.3) we have

$$d(w-v)_{yy} + f_k(w) - f_k(v) = (w-v)_t, \quad 0 < y < \rho, \quad t > 0. \tag{5.4}$$

Using (5.4), if $z(y,t) = (w(y,t) - v(y,t))e^{-\delta t}$ then z satisfies

$$de^{\delta t}z_{yy} + f_k(w) - f_k(v) = z_t e^{\delta t} + z\delta e^{\delta t}, \quad 0 < y < \rho, \quad t > 0. \tag{5.5}$$

Multiplying (5.5) by $e^{-\delta t}$ yields

$$dz_{yy} = z_t + \delta z + (f_k(v) - f_k(w))e^{-\delta t}. \tag{5.6}$$

Further, $z(0,t) = z(\rho,t) = 0$ for $t > 0$ and $z(y,0) = 0$ for $0 < y < \rho$. If $\inf\limits_{\substack{0<y<\rho \\ t>0}} \{z\} = z(y_o,t_o) < 0$ then by (5.6) [since $z < 0 \Rightarrow w < v \Rightarrow f_k(w) \geq f_k(v)$]

$$dz_{yy}(y_o,t_o) \leq \delta z(y_o,t_o) + z_t(y_o,t_o) \leq \delta z(y_o,t_o) < 0$$

and we have a contradiction. Thus $z \geq 0$ which implies $w - v \geq 0$. By a similar argument, the maximum principle also implies that $w - v \leq 0$ and we infer that $w = v$. Since w is a reflection of $v(x,t)$ for $0 \leq x \leq \rho$, the proof of the lemma is complete.

It is obvious from Lemma 5.1 that $v_x(0,t) = 0$ for all $t > 0$. Further, since $x = 0$ is interior to $[-\rho,\rho]$, we know that v_{xx} is continuous at $(0,t)$ for all $t > 0$. Also, by uniqueness, $v(x,t)$ restricted to $[0,\rho] \times [0,\infty)$ is precisely the solution to (2.14) with $\alpha = 0$. Finally, we note that for the solution $\{u,\gamma\}$ to (2.1), u_{xx} is the uniform limit of functions which are continuous at $x = 0$. Thus we conclude that $u_{xx}(0,t) = u_{xx}(0^+,t)$ exists

for all $t > 0$ where $u_{xx}(0^+,t) = \lim_{\varepsilon \to 0^+} u_{xx}(\varepsilon,t)$ for each $t > 0$. This leads to an important result in the following theorem.

Theorem 5.2. Let $\alpha = 0$ and in (2.1) let $\phi \in \mathcal{D}^-$. Then there exists a positive number T such that $0 < T < \infty$, $\gamma(T) = 0$, and such that $u(\cdot,t;\,\phi) = S^o(t)\phi \equiv 0$ for $t \geq T$.

Proof. From Proposition 3.12, we know $S^o(t) : \mathcal{D}^- \to \mathcal{D}^-$. Since $u_x(0,t) = 0$ for all $t > 0$ and since $u_{xx}(\cdot,t)$ is continuous on $[0,\gamma(t))$, we have that $u_{xx}(0,t) \leq 0$. Then $u_t(0,t) = u_{xx}(0,t) + f(u(0,t)) \leq f(u(0,t)) \leq f(0)$ for all $t > 0$. Since $f(0) = -\beta < 0$ and since $u(0,t)$ is finite, we conclude that $u(0,t)$ reaches zero in finite time. Thus we are assured of the existence of $T > 0$ as stated in the theorem. Since $u(\cdot,t) \in \mathcal{D}^-$, if $u(0,T) = 0$ and if θ is the zero function, then $u(\cdot,T) = \theta$. Finally, by Proposition 4.13 [since θ satisfies (4.18)], we know that $S^o(t)\phi \leq S^o(T)\phi = \theta$ for $t > T$. Since $S^o(t)\phi \in \mathcal{D}^-$ for all t, we conclude that $S^o(t)\phi = \theta$ for all $t \geq T$. This completes the proof of the theorem.

Corollary 5.3. For $\alpha = 0$ and $\phi \in \hat{\mathcal{D}}$, the trivial solution is a unique critical point for problem (2.1).

Proof. In the proof of Theorem 5.2, we showed that θ is the only critical point in $\mathcal{D}^-$ for problem (2.1). If $\psi \in \hat{\mathcal{D}}$ is any other critical point for (2.1) with $\alpha = 0$, then we have by (Q3) of Theorem 2.1 that

$$|\psi - \theta|_2 = |S^o(t)\psi - S^o(t)\theta|_2 \leq |\psi - \theta|_2 e^{-k_o t} < |\psi - \theta|_2 \tag{5.7}$$

whenever $t > 0$. But this is a contradiction unless $|\psi|_2 = 0$. Therefore $\psi = \theta$ and so θ is the unique critical point in $\hat{\mathcal{D}}$ for problem (2.1) (with $\alpha = 0$).

Having shown that the moving boundary $\gamma(t)$ reaches zero in finite time for $\alpha = 0$, we now proceed with similar discussion for the case $0 < \alpha < \infty$. We first prove a comparison result.

Lemma 5.4. Suppose $\Phi \in \mathcal{D}^-$ and let $(u^o(x,t),\gamma^o(t))$ be the solution of (2.1) for $\alpha = 0$ and initial value Φ. Similarly, let $(u^1(x,t),\gamma^1(t))$ be the solution of (2.1) for any fixed α satisfying $0 < \alpha < \infty$ and for initial value Φ. Then $u^1 \leq u^o$ for $(x,t) \in [0,\rho] \times [0,\infty)$.

Proof. For each $k \geq 1$, let $w_k^o(x,t)$ be the solution of (2.14) for $\alpha = 0$ and let $w_k^1(x,t)$ be the solution of (2.14) for the same α corresponding to $u^1(x,t)$. We will show that $w_k^1 \leq w_k^o$ for each integer $k \geq 1$. Since $\{w_k^1\}$ converges monotonically ($\downarrow$) to u^1 and since $\{w_k^o\}$ converges monotonically ($\downarrow$) to u^o, the lemma will be proved.

Suppose (for contradiction) that $\sup\limits_{\substack{0\leq x\leq\rho \\ t\geq 0}} \{w_k^1 e^{-\delta t} - w_k^o e^{-\delta t}\}$ $= e^{-\delta\hat{t}}(w_k^1(\hat{x},\hat{t}) - w_k^o(\hat{x},\hat{t})) > 0$ where $\delta > 0$ is a constant. Subtracting the appropriate differential equations of (2.14) yields

$$d(w_k^1 - w_k^o)_{xx} + f_k(w_k^1) - f_k(w_k^o) = (w_k^1 - w_k^o)_t, \quad 0 < x < \rho, \quad t > 0. \tag{5.8}$$

If we let $z(x,t) = e^{-\delta t}(w_k^1(x,t) - w_k^o(x,t))$ for $x \in [0,\rho]$, $t \geq 0$, then substitution in (5.8) gives

$$de^{\delta t}z_{xx} + f_k(w_k^1) - f_k(w_k^o) = e^{\delta t}z_t + \delta e^{\delta t}z. \tag{5.9}$$

Since $z(\rho,t) = e^{-\delta t}(w_k^1(\rho,t) - w_k^o(\rho,t)) = 0$, we have $\hat{x} < \rho$. Also $z_x(0,t) = e^{-\delta t}(\partial_x w_k^1(0,t) - \partial_x w_k^o(0,t)) = e^{-\delta t}(\partial_x w_k^1(0,t)) > 0$ and so $\hat{x} \neq 0$. Thus $\hat{x} \in (0,\rho)$ and we have

$$d z_{xx}(\hat{x},\hat{t}) = e^{-\delta t}[f_k(w_k{}^o(\hat{x},\hat{t})) - f_k(w_k{}^1(\hat{x},\hat{t}))]$$
$$z_t(\hat{x},\hat{t}) + \delta z(\hat{x},\hat{t}). \tag{5.10}$$

By hypothesis, $w_k{}^1(\hat{x},\hat{t}) > w_k{}^o(\hat{x},\hat{t})$ which implies $f_k(w_k{}^1(\hat{x},\hat{t})) \leq f_k(w_k{}^o(\hat{x},\hat{t}))$. Therefore $d z_{xx}(\hat{x},\hat{t}) \geq z_t(\hat{x},\hat{t}) + \delta z(\hat{x},\hat{t}) \geq \delta z(\hat{x},\hat{t}) > 0$ which is a contradiction. It must be the case then that $w_k{}^1 \leq w_k{}^o$ on $[0,\rho] \times [0,\infty)$ and the proof is complete.

Proposition 5.5. Let α be fixed and satisfy $0 < \alpha < \infty$. Then the moving boundary $\gamma(t)$ occurring in problem (2.1) with $\Phi \in \hat{\mathcal{D}}$ reaches zero in finite time, that is, there exists $T_1 > 0$ such that $\gamma(T_1) = 0$ and $u^1(\cdot,t;\ \Phi) = \theta$ for $t \geq T_1$.

Proof. Let $\Phi_o(x) \equiv b_o$ for $0 \leq x \leq \rho$. Using (Q4) of Theorem 2.1, we have that $u^1(x,t;\ \Phi) \leq u^1(x,t;\ \Phi_o)$ for all $\Phi \in \hat{\mathcal{D}}$. From Theorem 5.2, if $\alpha = 0$ then there exists $T_o > 0$ such that $u^o(\cdot,t;\ \Phi_o) = \theta$ for $t \geq T_o > 0$. Finally, by Lemma 5.4, $u^1(x,t;\ \Phi_o) \leq u^o(x,t;\ \Phi_o)$ for $t \geq 0$. Thus there exists $T_1 > 0$ such that $T_1 \leq T_o$ and such that $u^1(\cdot,t;\ \Phi) = \theta$ for $t \geq T_1$. This completes the proof of the proposition.

Before leaving the discussion of critical points for $\alpha < \infty$, we provide a time-decay result which is independent of initial value. For $0 < \alpha < \infty$, it is obviously not generally true that $u(x,t)$ satisfies $u_t \leq 0$. However, we can prove a "weaker" result in the following proposition.

Proposition 5.6. Suppose that $0 < \alpha < \infty$ in (2.1) and that (for $t_1 > 0$) $u(\cdot,t_1) \neq \theta$. Then, for $t_1 < t_2$, $\sup_{0\leq x\leq\rho} \{u(x,t_2)\} < \sup_{0\leq x\leq\rho} \{u(x,t_1)\}$.

Proof. Once again we make use of the (strong) maximum principle. Suppose the proposition is not true, so that for $0 < t_1 < t_2$, $\sup_{x\in[0,\rho]} \{u(x,t_2)\}$

$\geq \sup_{x \in [0,\rho]} \{u(x,t_1)\}$. Since $u_x(0,t_1) > 0$ by hypothesis, we know the supremums occur in $(0,\rho)$. Let $\sup_{\substack{0<x<\rho \\ t_1<t\leq t_2}} \{u(x,t)\} = u(\bar{x},\bar{t}) \geq \sup_{0<x<\rho} \{u(x,t_1)\}$.

By Theorem 2 of the appendix, $u(x,t) \equiv u(\bar{x},\bar{t})$ for all pairs (x,t) in $\left(0,\bar{x} + \frac{\gamma(\bar{t}) - \bar{x}}{2}\right) \times (\bar{t} - \delta,\bar{t})$ where δ is chosen small enough (by continuity of γ) so that $|\gamma(t) - \gamma(\bar{t})| \leq \frac{\gamma(\bar{t}) - \bar{x}}{4}$ when $\bar{t} - \delta < t < \bar{t}$. This is an obvious contradiction and the proposition is proved.

Remark. The strict inequality in the conclusion of Proposition 5.6 is another way of determining that θ is a unique critical point for (2.1) with $0 < \alpha < \infty$. Note that Proposition 5.6 is not sufficient by itself to prove that $\gamma(t)$ reaches zero in finite time.

At this juncture, we are finished with the cases where $0 \leq \alpha < \infty$ and the trivial solution is the critical point. In the remainder of this chapter, we restrict our attention to the case $\alpha = \infty$ where the critical point is non-trivial. First, we prove an inequality which is a bound for the critical point (steady-state) in the case $\alpha = \infty$.

Lemma 5.7. Suppose $\psi(x)$ is the steady-state solution for (2.1) with $\alpha = \infty$ (if one exists). Also suppose $f(\xi) < f(0)$ for $\xi > 0$. If $\sigma = (2c_o/K)^{\frac{1}{2}}$ where $K = -d^{-1}f(0)$, then

$$\psi(x) \leq P(x) \equiv (K/2)(\sigma - x)^2 \quad \text{for } x \in [0,\sigma]. \tag{5.11}$$

Proof. *Suppose (for contradiction) that*

$$\sup_{0\leq x\leq \sigma} \{\psi(x) - P(x)\} = \psi(x_o) - P(x_o) > 0.$$

Since $P''(x) = K$, we have for $x_o < \sigma$ that

$$d(\psi - P)''(x_o) = -f(\psi(x_o)) - dK > 0$$

since $\psi(x_o) > 0$. This contradicts the maximum principle. On the other hand, if $x_o = \sigma$ then $\psi(x_o)$ is positive and

$$\lim_{x \to \sigma^-} (\psi'(x) - P'(x)) = \psi'(\sigma^-).$$

But since $\psi(x_o) > 0$, we have $\psi'(\sigma^-) < 0$ by the remark following Proposition 3.12. Again we have a contradiction and, since $\psi(0) = P(0) = c_o$, we conclude that (5.11) is valid.

In a similar manner, we can find a lower bound for $\psi(x)$.

Lemma 5.8. Suppose that a steady-state solution $\psi(x)$ exists for (2.1) with $\alpha = \infty$. Also suppose that if $\xi < b_o$ then $f(b_o) < f(\xi)$. If $K_1 = -d^{-1}f(b_o)$ and if $\sigma_1 = (2c_o/K_1)^{\frac{1}{2}}$ then

$$R(x) \equiv (K_1/2)(\sigma_1 - x)^2 \leq \psi(x) \quad \text{for } x \in [0,\sigma_1]. \tag{5.12}$$

Proof. Suppose (for contradiction) that

$$\sup_{0 \leq x \leq \sigma_1} \{R(x) - \psi(x)\} = R(x_o) - \psi(x_o) > 0.$$

Since $R(\sigma_1) = 0$ and ψ is nonnegative, x_o must lie in $(0,\sigma_1)$. But in $(0,\sigma_1)$ a contradiction to the maximum principle occurs just as in the proof of Lemma 5. Thus (5.12) is valid.

The bound in Lemma 5.8 is helpful because it guarantees that, if a steady state $\psi(x)$ exists, then $\psi(x)$ is non-trivial. Lemma 5.7 provides a useful bound since for any initial value $\phi \in \mathcal{D}^-$ satisfying $\phi(x) \geq P(x)$, we will show later that $u(x,t;\phi) \geq \psi(x)$ for $t > 0$. It is now appropriate to state and prove the existence of such a critical point.

Theorem 5.9. For problem (2.1) with $\alpha = \infty$ there exists a unique steady-state solution $\psi(x)$ in $\mathcal{D}^-$ and there exists $\eta \in (0,\rho]$ such that $\psi(x)$ is twice continuously differentiable on $(0,\eta)$ and satisfies

$$\left.\begin{aligned} & d\psi''(x)+f(\psi(x)) = 0, \quad 0 < x < \eta \\ & \psi(0) = c_o, \quad \psi(\eta) = 0, \quad \psi(x) > 0 \quad \text{for } 0 < x < \eta \\ & \psi'(\eta^-) = 0, \quad \psi(x) = 0 \quad \text{for } \eta < x < \rho \end{aligned}\right\} \tag{5.13}$$

Additionally,

$$|S^\infty(t)\Phi - \psi|_2 \le |\Phi - \psi|_2 e^{-k_\infty t} \quad \text{for } t \ge 0, \quad \Phi \in \mathcal{D}^-. \tag{5.14}$$

Finally, if $\{\underline{u},\underline{\gamma}\}$ is the solution of (2.1) for initial value $\underline{\Phi} = \theta$ and if $\{\overline{u},\overline{\gamma}\}$ is the solution of (2.1) for initial value $\overline{\Phi}(x) \equiv b_o$, then

$$\lim_{t\to\infty} \overline{u}(x,t) = \lim_{t\to\infty} \underline{u}(x,t) = \psi(x) \tag{5.15}$$

uniformly on $[0,\rho]$.

Proof. Note first that if $\Phi \in \mathcal{D}^-$, then $\underline{\Phi} \le \Phi \le \overline{\Phi}$. Therefore since $S^\infty(t)\underline{\Phi} \in \mathcal{D}^-$, we know that $S^\infty(t)\underline{\Phi} \ge \underline{\Phi}$ for all $t \ge 0$, and so by the order preserving property (Q4) of Theorem 2.1 and by the semigroup property (Q2) (same theorem)

$$S^\infty(s + t)\underline{\Phi} = S^\infty(s)S^\infty(t)\underline{\Phi} \ge S^\infty(s)\underline{\Phi} \quad \text{for all } t,s \ge 0.$$

Thus $t \to S^\infty(t)\underline{\Phi}$ is increasing (and bounded above by $\overline{\Phi}$), so that

$$\underline{\psi} = \lim_{t\to\infty} S^\infty(t)\underline{\Phi}$$

exists in L^2. Further $\underline{\psi} \in \mathcal{D}^-$ by the L^2 continuity of S^∞ and

$$S^{\infty}(t)\underline{\psi} = S^{\infty}(t) \lim_{s\to\infty} S^{\infty}(s)\underline{\phi} = \lim_{s\to\infty} S^{\infty}(t)S^{\infty}(s)\underline{\phi} = \lim_{s\to\infty} S^{\infty}(t+s)\underline{\phi} = \underline{\psi}.$$

Obviously, $\underline{\psi}$ is a steady-state solution of (2.1) and since

$$|S^{\infty}(t)\phi - \underline{\psi}|_2 = |S^{\infty}(t)\phi - S^{\infty}(t)\underline{\psi}_{\,2}| \le |\phi - \underline{\psi}|_2 e^{-k_{\infty}t},$$

we have that (5.14) is valid. Uniqueness of ψ follows immediately using (5.14). In a similar manner, it follows that $t \to S^{\infty}(t)\overline{\Phi}$ is decreasing on $[0,\infty)$ and hence

$$\overline{\psi} = \lim_{t\to\infty} S^{\infty}(t)\overline{\Phi}$$

exists. Since $\overline{\psi}$ is also a critical point, we have immediately from (Q3) of Theorem 2.1 that $\underline{\psi} = \overline{\psi} = \psi$. Further, $\psi = S^{\infty}(t)\psi$ is continuously differentiable for $t > 0$ [see (2.17)] so it follows from Dini's Theorem that the limits in (5.15) are uniform on $[0,\rho]$. Finally, since $u(x,t;\psi) \equiv [S^{\infty}(t)\psi](x) \equiv \psi(x)$ is a solution of (2.1) that is time independent, it follows from Theorem 2.1 that (5.13) is satisfied by $\psi(x)$. This completes the proof of the theorem.

Since Theorem 5.9 guarantees existence of a unique non-trivial critical point for (2.1) when $\alpha = \infty$, we will now proceed with analysis to show that this critical point $\psi(x)$ cannot be reached in finite time.

Lemma 5.10. Let the initial value in (2.1) be $\Phi(x) = \overline{\psi}(x) \ge \psi(x)$ and let $\{\overline{u},\overline{\gamma}\} = \{u(x,t;\overline{\psi}),\gamma(t)\}$ be the corresponding solution pair of (2.1). Then $\overline{u}(x,t) \ge \psi(x)$ for all $t \ge 0$ and $x \in [0,\rho]$.

Proof. As in many of the other proofs we introduce the factor $e^{-\delta t}$ where $\delta > 0$. If $z(x,t) = e^{-\delta t}\psi(x)$ and if $v^k(x,t) = e^{-\delta t}\overline{u}^k(x,t)$ where $\overline{u}^k(x,t) \equiv [S_k^{\infty}(t)\overline{\psi}](x)$ then

$$0 = \psi_t = e^{\delta t} z_t + \delta e^{\delta t} z$$

and

$$\bar{u}_t{}^k = e^{\sigma t} v_t{}^k + \delta e^{\delta t} v^k.$$

After substituting into the differential equations of (2.1) and (2.14) we have

$$de^{\delta t} z_{xx} + f(e^{\delta t} z) = e^{\delta t} z_t + \delta e^{\delta t} z, \quad 0 < x < \eta, \quad t > 0 \qquad (5.16)$$

and

$$de^{\delta t} v_{xx}{}^k + f_k(e^{\delta t} v^k) = e^{\delta t} v_t{}^k + \delta e^{\delta t} v^k, \quad 0 < x < \rho, \quad t > 0 \qquad (5.17)$$

We seek a contradiction by supposing that $\sup\limits_{\substack{x\in[0,\eta]\\ t>0}} \{z - v^k\} > 0$. Subtracting (5.17) from (5.16) gives

$$d(z - v^k)_{xx} e^{\delta t} + f(e^{\delta t} z) - f_k(e^{\delta t} v^k) = \delta e^{\delta t}(z - v^k) + e^{\delta t}(z - v^k)_t.$$

At the maximum we have $(z - v^k)_t \geq 0$ so that (after multiplying by $e^{-\delta t}$)

$$d(z - v^k)_{xx} \geq [f_k(e^{\delta t} v^k) - f(e^{\delta t} z)]e^{-\delta t} + \delta(z - v^k).$$

Also, if $z \geq v^k$ then $e^{\delta t} z \geq e^{\delta t} v^k$ and $f(e^{\delta t} z) \leq f(e^{\delta t} v^k) \leq f_k(e^{\delta t} v^k)$. Thus at the maximum we have

$$d(z - v^k)_{xx} \geq \delta(z - v^k) > 0$$

which contradicts the maximum principle. Note that a positive maximum cannot occur at $x = \eta$ since $z(\eta,\cdot) = 0$ [assuming the support of $\psi(x)$ is $(0,\eta)$]. Also $z(0,t) = e^{-\delta t}\psi(0) = c_o e^{-\delta t} = e^{-\delta t}\bar{u}^k(0,t) = v^k(0,t)$. Thus we know that $v^k \geq z$ which implies $\bar{u}^k \geq \psi$ on $[0,\rho] \times [0,\infty)$. Since this is

true for all integers k, we conclude that $\bar{u}(x,t) \geq \psi(x)$ on $[0,\rho] \times [0,\infty)$. This completes the proof of the lemma.

By using a proof similar to that of Lemma 5.10, we observe that if $\underline{\psi}(x) \leq \psi(x)$ and $\{\underline{u},\underline{\gamma}\}$ is the solution of (2.1) corresponding to $\Phi(x) = \underline{\psi}(x)$, then $\underline{u}(x,t) \leq \psi(x)$ for all $t \geq 0$. An even stronger result is due to the next proposition.

<u>Proposition 5.11</u>. Suppose that $\underline{\psi}(x) \leq \psi(x)$ where $\psi(x)$ is the steady-state solution of (2.1) for $\alpha = \infty$. If $\{\underline{u}(x,t),\underline{\gamma}(t)\}$ is the solution of (2.1) corresponding to initial value $\Phi(x) = \underline{\psi}(x)$, then $\underline{u}(x,t)$ cannot reach $\psi(x)$ in finite time.

<u>Proof</u>. Suppose that $\underline{u}(x,T) = \psi(x)$, $0 \leq x \leq \eta$, at some time $t = T < \infty$. Assuming $(0,\eta)$ is the support of ψ, let $\hat{x} \in (0,\eta)$ and consider the solution of

$$\begin{aligned} & dv_{xx} + f(v) = v_t, \quad 0 < x < \hat{x}, \quad t > 0 \\ & v(0,t) = c_o, \quad v(\hat{x},t) = \underline{u}(\hat{x},t), \quad t > 0 \\ & \underline{\psi}(x) < v(x,0) < \psi(x), \quad 0 < x < \hat{x} \end{aligned} \tag{5.18}$$

Then by construction, $\underline{u}(x,t) \leq v(x,t) \leq \psi(x)$ for $0 \leq x \leq \hat{x}$ and $t \geq 0$. Further $v(0,t) = \underline{u}(0,t)$ and $v(\hat{x},t) = \underline{u}(\hat{x},t)$ for $t > 0$. By hypothesis $\underline{u}(x,T) = \psi(x)$ for $0 \leq x \leq \hat{x}$ and so $v(x,T) = \underline{u}(x,T) = \psi(x)$ for $0 \leq x \leq \hat{x}$. Now by backward uniqueness (see Theorem 4 of the appendix), we have that

$$v(x,t) = \underline{u}(x,t) \quad \text{for all } t < T.$$

This is a contradiction since $v(x,0) \neq \underline{\psi}(x)$ in (5.18). Thus $\underline{u}$ cannot attain ψ for any finite time and the proposition is proved.

With a similar proof based on backward uniqueness we have a parallel result for $\overline{u}(x,t)$.

Corollary 5.12. Suppose $\{\overline{u}(x,t),\overline{\gamma}(t)\}$ is the solution of (2.1) corresponding to the initial value $\Phi(x) = \overline{\psi}(x)$ satisfying $\psi(x) \leq \overline{\psi}(x)$ on $(0,\eta)$. Then $\overline{u}(x,t)$ does not reach $\psi(x)$ in finite time.

Note that Proposition 5.11 and Corollary 5.12 do not conclude anything about the moving boundary. Assuming $(0,\eta)$ is the support of the critical point ψ, the moving boundary might attain the value η in finite time. It turns out that this is not possible under most conditions. However, first we will use the (strong) maximum principle to prove the following:

Lemma 5.13. Suppose $\{\underline{u}(x,t),\underline{\gamma}(t)\}$ is the solution of (2.1) as given in Proposition 5.11. If $(0,\eta)$ is the support of $\psi(x)$, $\underline{u}(x,t) < \psi(x)$ for $x \in (0,\eta)$ and $t > 0$.

Proof. We seek a contradiction by assuming there exists $x_o \in (0,\eta)$ such that $\underline{u}(x_o,t_o) = \psi(x_o)$ at some time t_o. Letting $w(x,t) = \underline{u}(x,t) - \psi(x)$, we already know that $w \leq 0$. Also, after combining differential equations we have

$$dw_{xx} - w_t + f(\underline{u}) - f(\psi) = 0 \quad \text{for } (x,t) \in (0,\eta) \times (0,\infty). \tag{5.19}$$

Let

$$g(x,t) = \begin{cases} [f(\underline{u}(x,t)) - f(\psi(x))]/[\underline{u}(x,t) - \psi(x)] & \text{when } \underline{u} < \psi \\ f'(\psi(x)) & \text{when } \underline{u} = \psi \end{cases}$$

and then (5.19) becomes

$$dw_{xx} - w_t + g(x,t)w = 0.$$

Since f is nonincreasing and continuously differentiable, g is nonpositive. By hypothesis, w reaches its maximum (zero) at (x_o,t_o). But then by Theorem 2 of the appendix, w is identically zero for all $t < t_o$. This obviously contradicts $\underline{\psi}(x) \neq \psi(x)$ and so we conclude that x_o does not exist and the lemma is proved.

Lemma 5.14. Suppose $\Phi(x) = \overline{\psi}(x) \geq \psi(x)$ and $\{\overline{u},\overline{\gamma}(t)\}$ is the solution of (2.1) corresponding to $\overline{\psi}(x)$. Then $\overline{u}(x,t) > \psi(x)$ for $x \in (0,\eta)$ and $t > 0$.

Proof. The proof is the same as the one given for Lemma 5.13.

In order to discuss the behavior of the moving boundary, we will continue to denote the moving boundary $\overline{\gamma}(t)$ associated with $\overline{u}(x,t)$ given in Lemma 5.14. Also we assume $\overline{\psi}(\eta) > 0$ where again $(0,\eta)$ is the support of the steady-state solution $\psi(x)$. This assures that $\rho > \eta$ [where $(0,\rho)$ is the support of the initial value $\overline{\psi}(x)$] and precludes the possibility of no moving boundary when $\rho = \eta$ and $\overline{\psi}(x)$ satisfies condition (4.18). We can now prove that $\overline{\gamma}(t)$ does not reach η in finite time.

Theorem 5.15. Suppose that $\overline{\psi} \in \mathcal{D}^-$ and that $\overline{\psi}(\eta) > 0$. If $\{\overline{u}(x,t),\overline{\gamma}(t)\}$ is the solution of (2.1) for initial value $\Phi(x) = \overline{\psi}(x) \geq \psi(x)$, then $\overline{\gamma}(t) \neq \eta$ for any finite value of time T.

Proof. Let

$$w(x,t) = \psi(x) - \overline{u}(x,t) \quad \text{for } x \in [0,\eta].$$

Recall from Lemma 5.14 that $w < 0$ for $(x,t) \in (0,\eta) \times (0,\infty)$. From the differential equations we have

$$dw_{xx} + f(\psi) - f(\bar{u}) = w_t \quad \text{for } (x,t) \in (0,\eta) \times (0,\infty), \tag{5.20}$$

and let

$$g(x,t) = \begin{cases} \dfrac{f(\psi(x)) - f(\bar{u}(x,t))}{\psi(x) - \bar{u}(x,t)} & \text{if } \psi(x) < \bar{u}(x,t) \\ f'(\psi(x)) & \text{if } \psi(x) = \bar{u}(x,t) \end{cases} .$$

Then (5.20) becomes

$$dw_{xx} - w_t + g(x,t)w = 0 \quad \text{for } (x,t) \in (0,\eta) \times (0,\infty) \tag{5.21}$$

where $g(x,t)$ is again nonpositive. We seek a contradiction by assuming there exists $T > 0$ such that $\bar{\gamma}(T) = \eta$. This implies

$$w(\eta,T) = \psi(\eta) - \bar{u}(\eta,T) = 0$$

and

$$w_x(\eta^-,T) = \psi'(\eta^-) - \bar{u}_x(\eta^-,T) = 0.$$

However from (5.21) the function $w(x,t)$ satisfies the hypothesis of Theorem 3 in the appendix and (since zero is a maximum value for w) it must be the case that $w_x(\eta,T) > 0$. Therefore a contradiction occurs and we conclude that $\bar{\gamma}(T) > \eta$. Thus $\bar{\gamma}(t) > \eta$ for all t and the theorem is proved.

Remark. Note that Theorem 5.15 is true whether $\bar{u}(x,t)$ satisfies $\bar{u}_t \leq 0$ or not. Any function $\bar{u}(x,t)$ which is above $\psi(x)$ may increase locally in the time direction, yet in all cases its associated moving boundary $\bar{\gamma}(t)$ cannot attain the value η in finite time. Also note from the proof of Theorem 5.15 that $\psi'(0) < \bar{u}_x(0,t)$ for $t > 0$ since $w(x,t)$ must satisfy $0 > w_x(0,t) = \psi'(0) - \bar{u}_x(0,t)$ for $t > 0$.

<u>Proposition 5.16</u>. Suppose that $\underline{\psi} \in \mathcal{D}^-$ and that $(0,\sigma)$ is the support of $\underline{\psi}$ where $\sigma < \eta$. Let $\{\underline{u}(x,t),\underline{\gamma}(t)\}$ be the solution of (2.1) for initial value $\Phi(x) = \underline{\psi}(x) \leq \psi(x)$ and suppose $\underline{u}_t(x,t) \geq 0$. Assuming $(0,\eta)$ is the support of the steady-state solution $\psi(x)$, then $\underline{\gamma}(t)$ cannot reach η in finite time.

<u>Proof</u>. Suppose (for contradiction) that at $t = T$ the moving boundary satisfies $\underline{\gamma}(T) = \eta$. Since $\underline{u}_t \geq 0$ by hypothesis, we have that $\underline{\gamma}(t) = \eta$ for all $t \geq T$. Let

$$v(x,t) = \underline{u}(x,t) - \psi(x) \quad \text{for } (x,t) \in [0,\eta] \times [T,\infty)$$

and from Lemma 5.13 we have that $v(\cdot,t) < 0$ on $(0,\eta)$. Subtracting differential equations yields

$$dv_{xx} + f(\underline{u}) - f(\psi) = v_t \quad \text{for } (x,t) \in (0,\eta) \times (T,\infty) \tag{5.22}$$

and let

$$g(x,t) = \begin{cases} \dfrac{f(\underline{u}(x,t)) - f(\psi(x))}{\underline{u}(x,t) - \psi(x)} & \text{if } \underline{u}(x,t) < \psi(x) \\ f'(\psi(x)) & \text{if } \underline{u}(x,t) = \psi(x). \end{cases}$$

Then (5.22) becomes

$$dv_{xx} - v_t + g(x,t)v = 0 \quad \text{for } (x,t) \in (0,\eta) \times (T,\infty) \tag{5.23}$$

where $g(x,t)$ is nonpositive by construction. Now $\underline{\gamma}(T) = \eta$ implies

$$v(\eta,2T) = \underline{u}(\eta,2T) - \psi(\eta) = 0$$

and

$$v_x(\eta^-,2T) = \underline{u}_x(\eta^-,2T) - \psi'(\eta^-) = 0.$$

But the function $v(x,t)$ which satisfies (5.23) also satisfies the hypothesis of Theorem 3 in the appendix and therefore it must be the case that $v_x(\eta,2T) > 0$. This is an obvious contradiction and we conclude that $\underline{\gamma}(T) < \eta$. Thus for $t \geq 0$, $\underline{\gamma}(t) < \eta$ and the proposition is proved.

Remark. If $\underline{\psi} = \theta$ (the zero function) then the hypothesis of Proposition 5.16 is satisfied and we know that the associated moving boundary $\underline{\gamma}(t)$ satisfies $\underline{\gamma}(0) = 0$ and $\underline{\gamma}(t) < \eta$ for all $t > 0$.

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APPENDIX

In this thesis there are numerous applications of the strong maximum principle and backward uniqueness for a parabolic partial differential equation. In order to consolidate use of these well-known results, we state them and reference proofs in this appendix.

First, we restate problem (1.4) of the thesis as

$$\begin{aligned} &dw_{xx} + f(w) = w_t, \quad 0 < x < \rho, \quad t > 0 \\ &w_x(0,t) = \alpha w(0,t), \quad w(\rho,t) = 0, \quad t > 0 \\ &w(x,0) = \Phi(x), \quad 0 < x < \rho \end{aligned} \tag{A.1}$$

Next, recall that $\hat{\mathcal{D}}$ is the following subset of $\mathcal{D}$:

$\hat{\mathcal{D}} = \{\phi \in L^2 | 0 \leq \phi(x) \leq b_o$ a.e. and there exists a real number $r = r(\phi)$ such that ϕ is nondecreasing on $[0,r]$ and nonincreasing on $[r,\rho]\}$.

Finally, by Proposition 1.3, we know that the solution $w(x,t)$ to (A.1) lies below the function $u(x,t)$ which is in the solution pair to the moving boundary problem (1.3). Therefore we will assume in the lemma to follow that $w(\bar{x},\tau) = 0$ for some $t = \tau$ and some $x = \bar{x} < \rho$ and that $w(\cdot,\tau)$ is nonpositive on $[\bar{x},\rho]$. We also presume $\Phi \in \hat{\mathcal{D}}$.

<u>Lemma 1</u>. Suppose $w(x,t)$ [the solution to (A.1)] is nonpositive for $x \in [\bar{x},\rho]$ when $t = \tau$. Then there exist $y_1 = y_1(\tau)$ and $y_2 = y_2(\tau)$ such that $0 \leq y_1 < \bar{x} < y_2 < \rho$ and such that $w(\cdot,\tau)$ is nondecreasing on $[0,y_1]$, nonincreasing on $[y_1,y_2]$ and nondecreasing on $[y_2,\rho]$.

Proof. We will prove the lemma for the case $0 < \alpha < \infty$ and the remainder of the proof follows in a similar manner. [Observe that if $\alpha = 0$ or $\alpha = \infty$, then $y_1 = 0$.] In this proof, we assume the domain of $f(\cdot)$ is extended to $(-\infty,\infty)$ in such a manner that $f'(\cdot)$ is Hölder continuous on compact subsets of $(-\infty,\infty)$ and that $f' \leq 0$ on $(-\infty,\infty)$. From semigroup theory, we know that

$$w(\cdot,\tau) = \lim_{n\to\infty} (I - \frac{\tau}{n} B)^{-n}\phi \tag{A.2}$$

where the operator B satisfies

$$[B\psi](x) = d\psi''(x) + f(\psi(x)) \quad \text{for} \quad \psi \in \mathrm{Dom}(B),$$

$$\mathrm{Dom}(B) = \{\phi \in L^2 : \phi \text{ and } \phi' \text{ are absolutely continuous,}$$

$$\phi'' \in L^2, \quad \phi'(0) = \alpha\phi(0) \quad \text{and} \quad \phi(\rho) = 0\}.$$

Note that if $T(x)$ is the differentiable solution to

$$dT''(x) + f(T(x)) = \lambda T(x) - \lambda\Phi(x), \quad 0 < x < \rho, \quad \lambda > 0$$

$$T'(0) = \alpha T(0), \quad T(\rho) = 0$$

then either $T \in \hat{\mathcal{D}}$ or there exist z_1 and z_2 satisfying $0 < z_1 < z_2 < \rho$ such that $T(x)$ is nondecreasing on $[0,z_1]$, $[z_2,\rho]$ and nonincreasing on $[z_1,z_2]$. To show this, if Φ is differentiable on $[0,\rho]$, nondecreasing on $[0,r]$, and nonincreasing on $[r,\rho]$ then we suppose $T \notin \hat{\mathcal{D}}$. Because (for $0 < \alpha < \infty$) $T'(0) > 0$ and $T(\rho) = 0$, the existence of $z_1 > 0$ is obvious. Suppose (for contradiction) that $T'(x)$ attains a (local) negative minimum at x_o satisfying $0 < z_1 < x_o \leq r$. Then

$$d(T')''(x_o) = [\lambda - f'(T(x_o))]T'(x_o) - \lambda\Phi'(x_o) < 0$$

and we have a contradiction. Thus a negative minimum for $T'(x)$ must occur in (r,ρ). Next suppose that $T'(x)$ attains a (local) positive maximum at $\bar{x} \in (r,\rho)$. Then

$$d(T')''(\tilde{x}) = [\lambda - f'(T(\tilde{x}))]T'(\tilde{x}) - \lambda\phi'(\tilde{x}) > 0$$

and we again have a contradiction. So we have that $T'(x)$ attains a positive maximum either at $x = \rho$ or at $x = 0$. Thus it must be the case that there exists z_2 in $(0,\rho)$ at which $T(x)$ achieves a negative minimum and such that $T(x)$ is nondecreasing on $[0,z_1]$, $[z_2,\rho]$ and nonincreasing on $[z_1,z_2]$. We will call $T(x)$ s-shaped because it satisfies the above property. Last, consider the solution $\chi(x)$ to

$$d\chi''(x) + f(\chi(x)) = \lambda\chi(x) - \lambda\phi(x), \quad 0 < x < \rho, \quad \lambda > 0$$

$$\chi'(0) = \alpha\chi(0), \quad \chi(\rho) = 0$$

where $\phi(x)$ is s-shaped on $[0,\rho]$. We will show $\chi(x)$ to be s-shaped and then the conclusion follows from (A.2). Suppose $x_1,x_2 \in [0,\rho]$ exist such that $0 < x_1 < x_2 < \rho$ and such that $\phi(x)$ is nondecreasing on $[0,x_1]$, $[x_2,\rho]$ and nonincreasing on $[x_1,x_2]$. Further, suppose $\phi'(0) = \alpha\phi(0) > 0$, $\phi(\rho) = 0$ and ϕ achieves a negative minimum at $x = x_2$. Using the same argument as above, $\chi'(x)$ cannot have a (local) negative minimum on $(0,x_1]$. Neither can a (local) negative minimum for χ' occur on $[x_2,\rho)$ nor can a (local) positive maximum for χ' occur on $[x_1,x_2]$. Therefore we conclude that $\chi(x)$ is also s-shaped and the proof is complete.

<u>Theorem 2</u>. Let $h = h(x,t)$ be a prescribed function and let E be a region in the x,t-plane (that is, $E \subset \mathbb{R}^2$). Suppose $h \leq 0$ and $u(x,t)$ is a solution of the inequality

$$du_{xx} - u_t + hu \geq 0$$

in the region E. If the maximum M of u is attained at an interior point (x_0,t_0) of E and if $M \geq 0$, then $u \equiv M$ on all line segments of E where t is constant and $t \leq t_0$.

Proof. See Theorem 2 (p. 168) and Theorem 4 (p. 172) of Protter and Weinberger in [8].

Theorem 3. Let the function h(x,t) and the region E be as described in Theorem 1 above. Suppose u(x,t) is a solution of the inequality

$$du_{xx} - u_t + hu \geq 0$$

in the region E. Further suppose that P is a point on the boundary ∂E where the maximum of u occurs and that the normal to ∂E at P is not parallel to the t-axis. Also suppose that at P a circle tangent to ∂E can be constructed whose interior lies entirely in E and such that $u < M$ in this interior. Then

$$\frac{\partial u}{\partial \nu} > 0 \quad \text{at} \quad P$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on the boundary of E.

Proof. See Theorem 3 (p. 170) and Theorem 4 (p. 17.) of Protter and Weinberger in [8].

Theorem 4. Let the differential operator L be defined by

$$Lu = du_{xx} - u_t$$

and suppose L satisfies

$$|Lu|^2 \leq c_1|u|^2 + c_2|u_x|^2.$$

Further, suppose the variable x lies in a bounded interval. Now if there exists $T < \infty$ such that

$$u(x,T) = 0 \quad \text{for} \quad x \in [x_1,x_2]$$

and such that

$$u(x_1,t) = u(x_2,t) = 0 \quad \text{for} \quad 0 \leq t \leq T$$

then $u \equiv 0$ in the region $[x_1,x_2] \times [0,T]$.

Proof. A proof of such backward uniqueness may be found in Friedman (see Theorem 6 (p. 173) of [3]).

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